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Classification of 5-Dimensional Space-Time with Parallel 3-Branes

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Abstract

If the fifth dimension is one-dimensional connected manifold, up to diffeomorphic, the only possible space-time will be $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$. And there exist two possibilities on cosmology constant: the cosmology constant is constant along the fifth dimension, and the cosmology constant is sectional constant along the fifth dimension. We construct the general models with parallel 3-branes on those kinds of the space-time and with constant/sectional constant cosmology constant, and point out that for compact fifth dimension, the sum of the brane tensions is zero, for non-compact fifth dimension, the sum of the brane tensions is positive. We assume the observable brane which includes our world should have positive tension, and obtain that in those general scenarios, the 5-dimensional GUT scale on each brane can be indentified as the 5-dimensional Planck scale, but, the 4-dimensional Planck scale is generated from the low 4-dimensional GUT scale exponentially in our world. We also give some simple models to show explicitly how to solve the gauge hierarchy problem.

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1 Introduction

Experiments at LEP and Tevatron have given the strong support to the Standard Model $SU(3)_c \times SU(2)_L \times U(1)$. However, the Standard Model has some unattractive features which may imply the new physics. One of these problems is that the gauge forces and the gravitational force are not unified. Another is the gauge hierarchy problem between the weak scale and the 4-dimensional Planck scale. Previously, two solutions to the gauge hierarchy problem have been proposed: one is the idea of the technicolor and compositeness which lacks calculability, and the other is the idea of supersymmetry.

More than one year ago, it was suggested that the large compactified extra dimensions may also be the solution to the gauge hierarchy problem [1]. Furthermore, about half year ago, Randall and Sundrum [2] proposed another scenario that the extra dimension is an orbifold, and the size of the extra dimension is not large but the 4-dimensional mass scale in the standard model is suppressed by an exponential factor from 5-dimensional mass scale. In addition, they suggested that the fifth dimension might be infinity [3], and there may exist only one brane with positive tension at the origin, but, there exists the gauge hierarchy problem. The remarkable aspect of the second scenario is that it gives rise to a localized graviton field. Combining those results, Lykken and Randall obtained the following physical picture [4]: the graviton is localized on Planck brane, we live on a brane separated from the Planck brane about 30 Planck lengths along the fifth dimension. On our brane, the mass scale in the Standard Model is suppressed exponentially, which gives the low energy scale. We generalized those scenarios and obtained the scenario with the following property [18]: the 4-dimensional Planck scale is generated from the low 5-dimensional Planck scale by an exponential hierarchy, and the mass scale in the Standard model, which is contained in the observable brane, is not rescaled. In short, recently, this kind of compactification or similar idea has attracted a lot of attentions [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. By the way, supergravity domain walls was discussed previously in the 4-dimensional space-time [32].

In the later model building, there exist several approachs: (I) The 3-branes are parallel along the fifth dimension and the observable brane is one of those branes: Oda constructed the model with space-time $M^4 \times S^1$ where M^4 is the four-dimensional Minkowski space, and cosmology constant is constant on S^1 [12]. However, his solution is not general. H. Hatanaka, M. Sakamoto, M. Tachibana, and K. Takenaga constructed the model with space-time $M^4 \times S^1$ and cosmology constant is sectional constant [30]. And we constructed the model with only parallel positive tension 3-branes on $M^4 \times R^1$ and $M^4 \times R^1/Z_2$ [19]. (II) The observable sector or our world live in the brane intersections/junctions [6, 27, 28]. (III) Many brane intersections/junctions, which might be thought to be the combination of the approach (I) and (II) [20, 31]. In this paper, we classify the models of the first approach, or in other words, we classify 5-dimensional spce-time with parallel 3-branes. From differential topology/manifold, up to diffeomorphic, there is only one connected non-compact one-dimensional manifold: R^1, there is only one connected non-compact one-dimensional manifold with boundary: H_1 or R^1/Z_2 (the equivalence class is $y \sim -y$), there is only one connected compact one-dimensional manifold S^1, and there is only one connected compact one-dimensional manifold with boundary S^1/Z_2 or I. Therefore, the space-time we consider are $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$, $M^4 \times S^1/Z_2$. Furthermore, there are two possibilities for the cosmology constant: the cosmology constant is constant along the fifth dimension, and the cosmology constant is sectional constant along the fifth dimension.

Before our discussion, we would like to explain our assumption and notation which are similar to those in [18, 19]. We assume that all the gauge forces are unified on each 3-brane if there exist gauge forces. The 5-dimensional GUT scale on i-th 3-brane $M5_{GUT}^{(i)}$ and the 5-dimensional Planck scale M_X are defined as the GUT scale and Planck scale in the 5-dimensional fundamental metric, respectively. The 4-dimensional GUT scale on i-th 3-brane $M_{GUT}^{(i)}$ and the 4-dimenisonal Planck scale M_{pl} are defined as the GUT scale and Planck scale in the 4-dimensional Minkowski Metric ($\eta_{\mu\nu}$). In order to avoid the gauge hierarchy problem between the weak scale and the 4-dimensional GUT scale M_{GUT} on the observable brane which includes our world, we assume the low energy unification [2]. The key ansatz is that the 5-dimensional GUT scale on each brane is equal to the 5-dimensional Planck scale. In addition, as we know, an object with negative tension can not be stable, although we might not need to worry about it if the anti-brane is orientable, so, we consider the model with positive tension 3-branes and negative tension 3-branes. Moreover, because the brane with negative brane tension is an anti-gravity world [16], we assume that the observable brane which includes our world has positive tension. By the way, there are positive energy objects, namely D-branes and NS-branes, that are well understood and on which gauge fields and matter fields can be localized so that the Standard Model fields can be placed there.

In this paper, we construct the general models with parallel 3-branes on $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$, and $M^4 \times S^1/Z_2$. In general, if the space-time is compact, the sum of the brane tensions is zero, if the space-time is non-compact, the sum of the brane tensions is positive if one requires that the four-dimensional Planck scale is finite. In addition, if the cosmology constant is constant along the fifth dimension, we can define the index of the model: the sum of the brane tensions divided by $6kM_X^3$ where k is defined in the following section. If the space-time is $M^4 \times R^1$, the index is $+1$, if space-time is $M^4 \times R^1/Z_2$, the index is $+\frac{1}{2}$, and if the fifth dimension is compact, the index is 0. Generally speaking, the cosmology constant might be sectional constant along the fifth dimension. Moreover, for any point in $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$, which is not belong to any brane and

[2]We will not explain why M_{GUT} can be at low energy scale here, but, it is possible if one considers additional particles which change the RGE running. Of course, proton decay might be the problem, but we do not disscuss this here. Moreover, using extra dimensions, low-scale gauge coupling unification can occur due to power-law running induced by the presence of Kaluza-Klein states [21, 33], and proton-decay constraints might be satisfied in this framework.

the section where the cosmology constant is zero, there is a neighborhood which is diffeomorphic to (or a slice of) AdS_5 space. It seems to us the gauge hierarchy problem can be solved easily in these scenarios. Assume we had $l+m+1$ brane with position $y_{-l} < y_{-l+1} < ... < y_{m-1} < y_m$, $\sigma(y)$, which is defined in the following sections, will have minimal value at the brane with positive tension. Without loss of generality, we assume $\sigma(y_j)$ is minimal. If $\sigma(y_i) - \sigma(y_j)$ is not small for all $i \neq j$, then, M_{pl}^2/M_X^3 will be proportional to $e^{-2\sigma(y_j)}$. And if the observable brane is at $y = y_{i_o}$, then the 4-dimensional GUT scale in our world will be proportional to $M_{pl} \times e^{-(\sigma(y_{i_o})-\sigma(y_j))}$ under some conditions, if $\sigma(y_{i_o}) - \sigma(y_j) = 34.5$ and 30, we will push the four-dimensional GUT scale in our world to the TeV and 10^5 GeV range, respectively. The 4-dimensional GUT scale on $j-th$ brane is the highest, but, it does not mean the $j-th$ brane has larger brane tension. Furthermore, if $\sigma(y_j) > 0$ [3] and the $j-th$ brane is the observable brane, we can also solve the gauge hierarchy problem, but, the 5-dimensional Planck scale will be very large. In short, the 5-dimensional GUT scale on each brane can be indentified as the 5-dimensional Planck scale, but, the 4-dimensional Planck scale is generated from the low 4-dimensional GUT scale exponentially in our world.

We also give some simple models to show how to solve the gauge hierarchy problem.

This paper is organized as this way, in section 2, we discuss the general models with constant cosmology constant for the space-time $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$. In section 3, we discuss the general models with sectional constant cosmology constant for the space-time $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$. The conclusion is given in section 4.

2 General Model with Constant Cosmology Constant

First, we would like to consider the cosmology constant is constant along the fifth dimension, The space-time $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$ will be discussed in the subsection 2.1, 2.2, 2.3, 2.4, respectively. By the way, generally speaking, for any point in $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$, which is not belong to any brane, there is a neighborhood which is diffeomorphic to (or a slice of) AdS_5 space.

2.1 Space-Time $M^4 \times R^1$

In this subsection, we would like to consider the fifth dimension is R^1. Assume we have m parallel 3-branes, and their fifth coordinates are: $-\infty < y_1 < ... < y_{m-1} <$

[3] In this paper, we just consider $\sigma(y_j) > 0$ with no consant term in $\sigma(y)$, i. e. $c = 0$.

$y_m < +\infty$. The 5-dimensional metric in these branes are:

$$g^{(i)}_{\mu\nu}(x^\mu) \equiv G_{\mu\nu}(x^\mu, y = y_i) \ , \tag{1}$$

where G_{AB} is the five-dimensional metric, and $A, B = \mu, y$ [4].

The classical Lagrangian is given by:

$$S \ = \ S_{gravity} + S_B \ , \tag{2}$$

$$S_{gravity} \ = \ \int d^4x \ dy \ \sqrt{-G}\{-\Lambda + \frac{1}{2}M_X^3 R\} \ , \tag{3}$$

$$S_B \ = \ \sum_{i=1}^{m} \int d^4x \sqrt{-g^{(i)}}\{\mathcal{L}_i - V_i\} \ , \tag{4}$$

where M_X is the 5-dimensional Planck scale, Λ is the cosmology constant, and V_i where $i = 1, ..., m$ is the brane tension.

The 5-dimensional Einstein equation for the above action is:

$$\begin{aligned}\sqrt{-G}\left(R_{AB} - \frac{1}{2}G_{AB}R\right) \ &= \ -\frac{1}{M_X^3}[\Lambda\sqrt{-G} \ G_{AB} + \\ &\sum_{i=1}^{m} V_i\sqrt{-g^{(i)}} \ g^{(i)}_{\mu\nu} \ \delta^\mu_M \delta^\nu_N \ \delta(y - y_i)] \ .\end{aligned} \tag{5}$$

Assuming that there exists a solution that respects 4-dimensional Poincare invariance in the x^μ-directions, one obtains the 5-dimensional metric:

$$ds^2 = e^{-2\sigma(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \ . \tag{6}$$

With this metric, the Einstein equation reduces to:

$$\sigma'^2 = \frac{-\Lambda}{6M_X^3} \ , \tag{7}$$

$$\sigma'' = \sum_{i=1}^{m} \frac{V_i}{3M_X^3}\delta(y - y_i) \ . \tag{8}$$

Now, we consider the general solutions. For m is odd, assuming that $m = 2n+1$ where $n \geq 0$, we obtain:

$$\sigma(y)_A = \sum_{i=1}^{2n+1} (-1)^{i+1} k|y - y_i| + c \ , \tag{9}$$

[4]We assume that $G_{\mu 5} = 0$ here. In addition, if the fifth dimesnion has Z_2 symmetry, i. e., the Lagrangian is invariant under the transformation $y \leftrightarrow -y$, then, $G_{\mu 5} = 0$.

$$\sigma(y)_B = \sum_{i=1}^{2n+1} (-1)^i k|y - y_i| + c \ . \tag{10}$$

For m is even, assuming that $m = 2n$ where $n \geq 1$, we obtain:

$$\sigma(y)_C = \sum_{i=1}^{2n} (-1)^{i+1} k|y - y_i| - ky + c \ , \tag{11}$$

$$\sigma(y)_D = \sum_{i=1}^{2n} (-1)^i k|y - y_i| + ky + c \ . \tag{12}$$

And one can easily obtain the cosmology constant:

$$\Lambda_i = -6kM_X^3 \ . \tag{13}$$

For solution A and C, the bane tensions are:

$$V_i = (-1)^{i+1} 6kM_X^3 \ , \tag{14}$$

and for solution B and D, the brane tensions are:

$$V_i = (-1)^i 6kM_X^3 \ . \tag{15}$$

For the cosmology constant is constant along the fifth dimension, we can define the index of these kinds of models: the sum of brane tensions divided by $6kM_X^3$. Therefore, solution A has index $+1$, solution B has index -1, the solution C and D have index 0. If one required that the 4-dimensional Planck scale is finite, only solution A is reasonable. Therefore, we obtain the five-dimensional metric:

$$ds^2 = e^{-2\sum_{i=1}^{2n+1} (-1)^{i+1} k|y-y_i| - 2c} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \ . \tag{16}$$

And the corresponding four-dimensional Planck scale is:

$$M_{pl}^2 = \frac{1}{k} M_X^3 \left(\sum_{i=1}^{2n+1} (-1)^{i+1} e^{-2\sigma(y_i)} \right) \ . \tag{17}$$

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} \ . \tag{18}$$

Our general assumption is that $M5_{GUT}^{(i)} \equiv M_X$, for $i = 1, ..., 2n + 1$.

This solution can be generalized to the solution with Z_2 symmetry: one just requires that, $y_{n+1} = 0$, $y_i = -y_{2n+2-i}$ for $1 \leq i \leq n$.

Now, we would like to give two simple examples.

(I) Three brane case: their positions are y_1, y_2, y_3, respectively, and the values of the brane tension divided by $6M_X^3$ are: k, $-k$, k, respectively. Therefore, we obtain:

$$\sigma(y) = \sum_{i=1}^{3} (-1)^{i+1} k|y - y_i| + c \ , \tag{19}$$

$$\sigma(y_1) = k(y_3 - y_2) + c \ , \ \ \sigma(y_2) = k(y_3 - y_1) + c \ , \tag{20}$$

$$\sigma(y_3) = k(y_2 - y_1) + c \ . \tag{21}$$

Without loss of generality, we assume $y_2 - y_1 < y_3 - y_2$. The four-dimensional Planck scale are

$$M_{pl}^2 = \frac{M_X^3}{k} e^{-2\sigma(y_3)} \left(e^{-2(\sigma(y_1)-\sigma(y_3))} - e^{-2(\sigma(y_2)-\sigma(y_3))} + 1 \right) \ . \tag{22}$$

If the brane with position y_1 is the observable brane which includes our world, we can solve the gauge hierarchy problem. Assuming that $e^{-2(\sigma(y_1)-\sigma(y_3))} << 1$, and $M_X = k$, we obtain:

$$M_{GUT}^{(1)} = M_{pl} e^{-k(y_3+y_1-2y_2)} \ . \tag{23}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $k(y_3 + y_1 - 2y_2) = 34.5$ and 30, respectively. And the value of $\sigma(y_3) = k(y_2 - y_1) + c$ determines the relation between M_{pl} and M_X.

If the brane with position y_3 is the observable brane which includes our world, we can solve the gauge hierarchy problem, too. Assuming that $M_{pl} = k$ and $e^{-2(\sigma(y_1)-\sigma(y_3))} << 1$, we obtain:

$$M_{GUT}^{(3)} = M_{pl} e^{-\frac{1}{3}k(y_2-y_1)-\frac{1}{3}c} \ , \tag{24}$$

with $k(y_2-y_1)+c = 103.5$ and 90, we can have GUT scale in our world at TeV scale and 10^5 GeV scale, respectively. Obviously, in this case, because of the exponential factor, the five dimensional Planck scale is very high, 10^{48} GeV and 10^{44} GeV, respectively.

(II) Five brane case with Z_2 symmetry: their positions are: $y_1 = -y_5$, $y_2 = -y_4$, $y_3 = 0$, y_4, y_5, respectively, and the values of the brane tensions divided by $6M_X^3$ are: k, $-k$, k, $-k$, k, respectively. Therefore, we obtain:

$$\sigma(y) = k|y| + \sum_{i=4}^{5} (-1)^{i+1} k(|y - y_i| + |y + y_i|) + c \ , \tag{25}$$

$$\sigma(y_1) = \sigma(y_5) = k y_5 + c \ , \tag{26}$$

$$\sigma(y_2) = \sigma(y_4) = 2k y_5 - k y_4 + c \ , \tag{27}$$

$$\sigma(y_3) = 2k(y_5 - y_4) + c \ . \tag{28}$$

Without loss of generality, we assume that, $y_5 > 2y_4$. The four-dimensional Planck scale are:

$$M_{pl}^2 = \frac{M_X^3}{k} e^{-2ky_5-2c} \left(2 - 2e^{-2k(y_5-y_4)} + e^{-2k(y_5-2y_4)} \right) \ . \tag{29}$$

If the brane with position y_3 is the observable brane, the gauge hierarchy problem can also be solved. Assuming that $e^{-2k(y_5-2y_4)} << 1$, and $M_X = k$, we obtain:

$$M_{GUT}^{(3)} = M_{pl} e^{-k(y_5-2y_4)} \ . \tag{30}$$

So, the GUT scale in our world will be pushed to TeV scale and 10^5 GeV scale range if $k(y_5 - 2y_4) = 34.5$ and 30, respectively. And the value of $ky_5 + c$ determines the relation between M_{pl} and M_X.

If the brane with position y_1 is the observable brane, one can also solve the gauge hierarchy problem. Assuming that $M_{pl} = k$ and $e^{-2k(y_5-2y_4)} << 1$, one obtains:

$$M_{GUT}^{(1)} = M_{pl} e^{-\frac{1}{3}ky_5 - \frac{1}{3}c} \ , \tag{31}$$

with $ky_5 + c = 103.5$ and 90, one obtains that the GUT scale in our world is at TeV scale and 10^5 GeV scale, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

2.2 Space-Time $M^4 \times R^1/Z_2$

Now, we consider the space-time is $M^4 \times R^1/Z_2$. The solution in subsection 2.1 can also be generalized to this case in which the fifth dimension is R^1/Z_2. If the solution in last subsection has Z_2 symmetry, we just introduce equivalence classes: $y \sim -y$ and $i - th\ brane \sim (2n+2-i) - th\ brane$. In fact, the only difference from above R^1 case is that the fifth dimension is H_1 or R^1/Z_2, i.e., $0 \leq y < +\infty$, so, the integration of dy is from 0 to ∞. The trick point is the tension of brane at origin. We would like to explain the technical detail as this way: the brane at origin which is $(n+1)$-th brane with tension $(-1)^n 6kM_X^3$, we split it into two branes with equal tension $(-1)^n 3kM_X^3$, and the positions $y = -\epsilon$ and $y = +\epsilon$, respectively. In the limit, $\epsilon \to 0$, we obtain the original case. Therefore, $\frac{d\sigma(y)}{dy}$ is zero at $y = 0$. And the brane tension V_{n+1} is half of the original value, i. e., $V_{n+1} = (-1)^n 3kM_X^3$ after module equivalence classes. In general, using this method, we obtain $\frac{d\sigma(y)}{dy}$ is zero at boundary. And in this case, the index is $\frac{1}{2}$. In short, the result is similar to that in last subsection with Z_2 symmetry.

For n is odd, we obtain:

$$\sigma(y) = \sum_{i=n+2}^{2n+1} (-1)^{i+1} k|y - y_i| + c' \ , \tag{32}$$

and for n is even, we obtain:

$$\sigma(y) = k|y| + \sum_{i=n+2}^{2n+1} (-1)^{i+1} k|y - y_i| + c' \ , \tag{33}$$

where

$$c' = c + \sum_{i=1}^{n} (-1)^{i+1} k y_{2n+2-i} \ . \tag{34}$$

Because c' is just a constant, we will write it as c in the following part. Similar convention will be used in the following subsections.

For the simplicity, we just renumber above subscript i where $i = n+1, ..., 2n+1$ as $i-(n+1)$. So, we have $n+1$ parallel 3-branes, with position $y_0 = 0 < y_1 < ... < y_{n-1} < y_n < +\infty$. The $\sigma(y)$ can be written as:

$$\sigma(y) = \frac{1}{2}(1+(-1)^n)k|y| + \sum_{i=1}^{n}(-1)^{i+n}k|y-y_i| + c \ . \tag{35}$$

One can easily obtain the cosmology constant

$$\Lambda = -6kM_X^3 \ , \tag{36}$$

and the bane tensions are:

$$V_0 = (-1)^n 3kM_X^3 \ , \ V_i = (-1)^{i+n} 6kM_X^3 \ . \tag{37}$$

And the corresponding 4-dimensional Planck scale is:

$$M_{pl}^2 = \frac{1}{k}M_X^3 \left(\frac{(-1)^n}{2} e^{-2\sigma(y_0)} + \sum_{i=1}^{n}(-1)^{i+n} e^{-2\sigma(y_i)} \right) \ . \tag{38}$$

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} \ . \tag{39}$$

Now, we discuss the example, for three brane case, it is equivalent to above 5-brane case, so, we donot rediscuss it here. We discuss four 3-brane case: their positions are y_0, y_1, y_2, y_3, respectively, and the values of the brane tensions divided by $6M_X^3$ are: $-\frac{1}{2}k$, k, $-k$, k, respectively. Therefore, we obtain:

$$\sigma(y) = \sum_{i=1}^{3}(-1)^{i+1}k|y-y_i| + c \ , \tag{40}$$

$$\sigma(y_0) = k(y_3 + y_1 - y_2) + c \ , \ \sigma(y_1) = k(y_3 - y_2) + c \ , \tag{41}$$

$$\sigma(y_2) = k(y_3 - y_1) + c \ , \ \sigma(y_3) = k(y_2 - y_1) + c \ . \tag{42}$$

Without loss of generality, we assume $y_2 - y_1 > y_3 - y_2$. The four-dimensional Planck scale are

$$M_{pl}^2 = \frac{M_X^3}{k} e^{-2k(y_3-y_2)-2c} \left(-\frac{1}{2} e^{-2ky_1} + 1 - e^{-2k(y_2-y_1)} + e^{-2k(2y_2-y_1-y_3)} \right) \ . \tag{43}$$

If the brane with position y_3 is the observable brane, we can solve the gauge hierarchy problem. Assuming that $e^{-2k(2y_2-y_1-y_3)} << 1$, and $M_X = k$, we obtain:

$$M_{GUT}^{(3)} = M_{pl} e^{-k(2y_2-y_1-y_3)} \ . \tag{44}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $k(2y_2 - y_1 - y_3) = 34.5$ and 30, respectively. And the value of $\sigma(y_1) = k(y_3 - y_2) + c$ determines the relation between M_{pl} and M_X.

If the brane with position y_1 is the observable brane, we can solve the gauge hierarchy problem, too. Assuming that $M_{pl} = k$ and $e^{-2k(2y_2-y_1-y_3)} << 1$, we obtain:

$$M_{GUT}^{(1)} = M_{pl} e^{-\frac{1}{3}k(y_3-y_2)-\frac{1}{3}c} , \tag{45}$$

with $k(y_3 - y_2) + c = 103.5$ and 90, we can have GUT scale in our world at TeV scale and 10^5 GeV scale, respectively. The five dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

2.3 Space-Time $M^4 \times S^1$

Now, we consider the fifth dimension is S^1. Because the adjacent two branes will have opposite brane tensions, i. e., one has positive tension, and the other one has negative tension, we will have even number of branes and the index is 0. Let us assume that we have $2n$ brane, and their position are $0 \leq y_1 < y_2 < ... < y_{2n-1} < y_{2n} < 2\pi\rho$, where ρ is the radius of the fifth dimension. The Lagrangian, Einstein equation, and the differential equations of $\sigma(y)$ are similar to those in subsection 2.1 except we must have even number of branes, and $0 \leq y \leq 2\pi\rho$, and $\sigma(0) = \sigma(2\pi\rho)$.

If $y_1 = 0$, then, the solutions of $\sigma(y)$ are

$$\sigma(y)_A = \sum_{i=2}^{2n} (-1)^{i+1} k|y - y_i| + c , \tag{46}$$

$$\sigma(y)_B = \sum_{i=2}^{2n} (-1)^{i} k|y - y_i| + c . \tag{47}$$

If $y_1 \neq 0$, then, the solutions are:

$$\sigma(y)_C = \sum_{i=1}^{2n} (-1)^{i+1} k|y - y_i| - ky + c , \tag{48}$$

$$\sigma(y)_D = \sum_{i=1}^{2n} (-1)^{i} k|y - y_i| + ky + c . \tag{49}$$

And one can easily obtain the cosmology constant

$$\Lambda_i = -6kM_X^3 . \tag{50}$$

For solution A and C, the bane tensions are:

$$V_i = (-1)^{i+1} 6kM_X^3 , \tag{51}$$

and for solution B and D, the brane tensions are:

$$V_i = (-1)^i 6kM_X^3 \ . \tag{52}$$

The constraint is $\sigma(0) = \sigma(2\pi R)$. So, we obtain the constraint equations on solutions A, B, C, D, respectively:

$$\sum_{i=2}^{2n}(-1)^{i+1} y_i = -L \ , \tag{53}$$

$$\sum_{i=2}^{2n}(-1)^{i} y_i = L \ , \tag{54}$$

$$\sum_{i=1}^{2n}(-1)^{i+1} y_i = -L \ , \tag{55}$$

$$\sum_{i=1}^{2n}(-1)^{i} y_i = L \ . \tag{56}$$

The corresponding 4-dimensional Planck scale for case A and C is:

$$M_{pl}^2 = \frac{1}{k} M_X^3 \left(\sum_{i=1}^{2n} (-1)^{i+1} e^{-2\sigma(y_i)_F} \right) \ , \tag{57}$$

where $F = A$, or D. And the corresponding 4-dimensional Planck scale for case B and D is:

$$M_{pl}^2 = \frac{1}{k} M_X^3 \left(\sum_{i=1}^{2n} (-1)^{i} e^{-2\sigma(y_i)_F} \right) \ , \tag{58}$$

where $F = B$ or D.

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)_F} \ , \tag{59}$$

where $F = A, B, C, D$.

In addition, we can consider the fifth dimension with Z_2 symmetry. The Z_2 symmetry is defined by the transformation of $y \leftrightarrow 2\pi\rho - y$, and the metric and Lagrangian is invariant under this transformation. With Z_2 symmetry, we must have a brane at position 0, and a brane at position $\pi\rho$. Therefore, only solution (A) and (B) are possible. So, $y_1 = 0$, $y_{n+1} = \pi\rho$, $y_i = 2\pi\rho - y_{2n+2-i}$ for $i = 2, ..., n$.

Let us give an explicit model. We discuss four 3-brane case: their positions are: $y_1 = 0$, y_2, y_3, y_4, respectively, and the values of the brane tensions divided by $6M_X^3$ are: k, $-k$, k, $-k$, respectively. Therefore, we obtain:

$$\sigma(y) = \sum_{i=2}^{4} (-1)^{i+1} k|y - y_i| + c \ , \tag{60}$$

so, the constraint equation is:

$$y_3 = y_2 + y_4 - \pi\rho \ . \tag{61}$$

And one can easily obtain:

$$\sigma(y_1) = -k\pi\rho + c \ , \ \sigma(y_2) = -k(\pi\rho - y_2) + c \ , \tag{62}$$

$$\sigma(y_3) = -k(y_4 - y_2) + c \ , \ \sigma(y_4) = -k(y_4 - \pi\rho) + c \ . \tag{63}$$

Without loss of generality, we assume $y_4 - y_2 < \pi\rho$. The four-dimensional Planck scale are

$$M_{pl}^2 = \frac{M_X^3}{k} e^{2\pi k\rho - 2c} \left(1 - e^{-2ky_2} + e^{-2k(\pi\rho - y_4 + y_2)} - e^{-2k(2\pi\rho - y_4)}\right) \ . \tag{64}$$

If the brane with position y_3 is the observable brane, we can solve the gauge hierarchy problem. Assuming that $e^{-2k(\pi\rho - y_4 + y_2)} << 1$ and $e^{-2ky_2} << 1$, and $M_X = k$, we obtain:

$$M_{GUT}^{(3)} = M_{pl} e^{-k(\pi\rho - y_4 + y_2)} \ . \tag{65}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $k(\pi\rho - y_4 + y_2) = 34.5$ and 30, respectively. And the value of $\pi\rho k - c$ determines the relation between M_{pl} and M_X.

2.4 Space-Time $M^4 \times S^1/Z_2$

Now, we would like to consider the space-time is $M^4 \times S^1/Z_2$. The solution with Z_2 symmetry in subsection 2.3 can be generalized to this case. For the solution with Z_2 symmetry in 2.3, we introduce the equivalence classes: $y \sim 2\pi\rho - y$, and $i-th\ brane \sim (2n+2-i)-th\ brane$. Moduling the equivalence classes, we obtain the model on $M^4 \times S^1/Z_2$ with $n+1$ brane. Using our splitting boundary branes method as before, we obtain the tensions of the boundary branes will be half of the original values. So, the index in this case is zero, too, which is the only constraints in this kind of model. In short, after module equivalence classes, we will have $n+1$ brane with position $y_0 = 0 < y_1 < ... < y_{n-1} < y_n = \pi\rho$.

The general solutions of $\sigma(y)$ are

$$\sigma(y) = sign(V_0) \left(\sum_{i=1}^{n-1} (-1)^i k|y - y_i| + \frac{1}{2}(1 + (-1)^{n+1})ky\right) + c \ , \tag{66}$$

where $sign(V_i) = |V_i|/V_i$, for $i = 1, ..., n$.

The corresponding 4-dimensional Planck scale is:

$$M_{pl}^2 = \frac{sign(V_0)}{k} M_X^3 \left(\frac{1}{2} e^{-2\sigma(y_0)} + \sum_{i=1}^{n-1} (-1)^i e^{-2\sigma(y_i)} + \frac{(-1)^n}{2} e^{-2\sigma(y_n)}\right) \ . \tag{67}$$

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} \ . \tag{68}$$

Here, we will give two simple examples.

(I) Although the three brane case is equivalent to above four 3-brane case with Z_2 symmetry module the equivalence classes. Here, we also discuss three 3-brane case, for its simplicity and phenomenology interesting. Assuming we have three brane with positions: $y_0 = 0$, y_1, $y_2 = \pi\rho$, respectively, and the values of the brane tension divided by $6M_X^3$: $\frac{1}{2}k$, $-k$, $\frac{1}{2}k$, respectively, we obtain:

$$\sigma(y) = -k|y - y_1| + c \ , \tag{69}$$

$$\sigma(y_0) = -ky_1 + c \ , \ \sigma(y_1) = c \ , \tag{70}$$

$$\sigma(y_2) = -k(\pi\rho - y_1) + c \ . \tag{71}$$

Without loss of generality, we assume $\pi\rho < 2y_1$. The four-dimensional Planck scale are

$$M_{pl}^2 = \frac{M_X^3}{2k} e^{2ky_1 - 2c} \left(1 - 2e^{-2ky_1} + e^{-2k(2y_1 - \pi\rho)}\right) \ . \tag{72}$$

If the brane with position y_2 is the observable brane, we can solve the gauge hierarchy problem. Assuming that $e^{-2k(2y_1-\rho)} << 1$, and $M_X = 2k$, we obtain:

$$M_{GUT}^{(2)} = M_{pl} e^{-k(2y_1 - \pi\rho)} \ . \tag{73}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $k(2y_1 - \pi\rho) = 34.5$ and 30, respectively. And the value of $ky_1 - c$ determines the relation between M_{pl} and M_X.

(II) We discuss four 3-brane case: their positions are: $y_0 = 0$, y_1, y_2, $y_3 = \pi\rho$, respectively, and the values of the brane tension divided by $6M_X^3$ are: $\frac{1}{2}k$, $-k$, k, $-\frac{1}{2}k$, respectively. Therefore, we obtain:

$$\sigma(y) = ky + \sum_{i=1}^{2} (-1)^i k|y - y_i| + c \ , \tag{74}$$

$$\sigma(y_0) = k(y_2 - y_1) + c \ , \ \sigma(y_1) = ky_2 + c \ , \tag{75}$$

$$\sigma(y_2) = ky_1 + c \ , \ \sigma(y_3) = k(\pi\rho - y_2 + y_1) + c \ . \tag{76}$$

Without loss of generality, we assume $2y_1 < y_2$. The four-dimensional Planck scale are

$$M_{pl}^2 = \frac{M_X^3}{k} e^{-2ky_1 - 2c} \left(\frac{1}{2} e^{-2k(y_2 - 2y_1)} - e^{-2k(y_2 - y_1)} + 1 - \frac{1}{2} e^{-2k(\pi\rho - y_2)}\right) \ . \tag{77}$$

If the brane with position y_0 is the observable brane, we can solve the gauge hierarchy problem. Assuming that $e^{-2k(y_2-2y_1)} << 1, e^{-2k(\pi\rho-y_2)} << 1$, and $M_X = k$, we obtain:

$$M_{GUT}^{(0)} = M_{pl} e^{-k(y_2-2y_1)} \ . \tag{78}$$

So, the GUT scale in our world will be pushed to TeV scale and 10^5 GeV scale range if $k(y_2-2y_1) = 34.5$ and 30, respectively. And the value of $\sigma(y_2) = ky_1 + c$ determines the relation between M_{pl} and M_X.

If the brane with position y_2 is the observable brane, one can also solve the gauge hierarchy problem. Assuming that $M_{pl} = k$ and $e^{-2k(y_2-2y_1)} << 1, e^{-2k(\pi\rho-y_2)} << 1$, one obtains:

$$M_{GUT}^{(2)} = M_{pl} e^{-\frac{1}{3}ky_1-\frac{1}{3}c} \ , \tag{79}$$

with $ky_1 + c = 103.5$ and 90, one will have the GUT scale in our world at TeV scale and 10^5 GeV scale range, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

3 General Model with Sectional Constant Cosmology Constant

Now, we consider the cosmology constant is not constant along the fifth dimesnion, although it is constant between any two adjacent branes, and between $+\infty/-\infty$ and the brane with maximum/minimum fifth dimension coordinate. The space-time $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$ will be discussed in the subsection 3.1, 3.2, 3.3, 3.4, respectively. In addition, we assume that $\chi_{i,i+1} \neq 0$ in the discussion of the explicit models, where $\chi_{i,i+1}$ is defined in the following subsections.

3.1 Space-Time $M^4 \times R^1$

First, we consider the fifth dimension is R^1. Assuming we have $l + m + 1$ parallel 3-branes, and their fifth coordinastes are: $-\infty < y_{-l} < y_{-l+1} < ... < y_{-1} < y_0 < y_1 < ... < y_{m-1} < y_m < +\infty$. The 5-dimensional metric in these branes are:

$$g_{\mu\nu}^{(i)}(x^\mu) \equiv G_{\mu\nu}(x^\mu, y = y_i) \ , \tag{80}$$

where G_{AB} is the five-dimensional metric, and $A, B = \mu, y$.

The classical Lagrangian is given by:

$$S = S_{gravity} + S_B \ , \tag{81}$$

$$S_{gravity} = \int d^4x \ dy \ \sqrt{-G}\{-\Lambda(y) + \frac{1}{2} M_X^3 R\} \ , \tag{82}$$

$$S_B = \sum_{i=-l}^{m} \int d^4x \sqrt{-g^{(i)}} \{\mathcal{L}_i - V_i\} \ , \tag{83}$$

where M_X is the 5-dimensional Planck scale, $\Lambda(y)$ is the cosmology constant, and V_i where $i = -l, ..., m$ is the brane tension. The $\Lambda(y)$ is defined as the following:

$$\begin{aligned}
\Lambda(y) \ &= \ \sum_{i=1}^{m} \Lambda_i \left(\theta(y - y_{i-1}) - \theta(y - y_i)\right) + \Lambda_{+\infty}\theta(y - y_m) \\
&+ \sum_{i=-l+1}^{0} \Lambda_i \left(\theta(-y + y_i) - \theta(-y + y_{i-1})\right) + \Lambda_{-\infty}\theta(-y + y_{-l}) \ ,
\end{aligned} \tag{84}$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. From advanced calculus, we know that $\Lambda(y)$ is piece-wise continuous or sectionally continuous, exactly speaking, $\Lambda(y)$ is sectional constant.

The 5-dimensional Einstein equation for the above action is:

$$\begin{aligned}
\sqrt{-G}\left(R_{AB} - \frac{1}{2}G_{AB}R\right) \ &= \ -\frac{1}{M_X^3}[\Lambda(y)\sqrt{-G} \ G_{AB} + \\
& \sum_{i=-l}^{m} V_i \sqrt{-g^{(i)}} \ g_{\mu\nu}^{(i)} \ \delta_M^\mu \delta_N^\nu \ \delta(y - y_i)] \ .
\end{aligned} \tag{85}$$

Assuming that there exists a solution that respects 4-dimensional Poincare invariance in the x^μ-directions, one obtains the 5-dimensional metric:

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \ . \tag{86}$$

With this metric, the Einstein equation reduces to:

$$\sigma'^2 = \frac{-\Lambda(y)}{6M_X^3} \ , \ \sigma'' = \sum_{i=-l}^{m} \frac{V_i}{3M_X^3} \delta(y - y_i) \ . \tag{87}$$

So, σ' is also sectionally continuous or sectional constant.

The general solution to above differential equations is:

$$\sigma(y) = \sum_{i=-l}^{m} k_i |y - y_i| + k_c y + c \ , \tag{88}$$

where k_c and c are constants, and $k_i \neq 0$ for $i = -l, ..., m$. The relations between the k_i and V_i, and the relations between the k_i and Λ_i are:

$$V_i = 6k_i M_X^3 \ , \tag{89}$$

$$\Lambda_i = -6M_X^3 (\sum_{j=i}^{m} k_j - \sum_{j=-l}^{i-1} k_j - k_c)^2 \ , \tag{90}$$

$$\Lambda_{-\infty} = -6M_X^3(\sum_{j=-l}^{m} k_j - k_c)^2 \ , \ \Lambda_{+\infty} = -6M_X^3(\sum_{j=-l}^{m} k_j + k_c)^2 \ . \tag{91}$$

Therefore, the cosmology constant is negative except the section between the two branes where the cosmology constant is zero, then, for any point in $M^4 \times R^1$, which is not belong to any brane and that kinds of sections, there is a neighborhood which is diffeomorphic to (or a slice of) AdS_5 space, similarly, this statement is correct for the space-time $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$. Moreover, the cosmology constant and brane tensions should satisfy above equations. In order to obtain finite 4-dimensional Planck scale, we obtain the constraints: $\sum_{j=-l}^{m} k_j > |k_c|$. So, the sum of brane tension is positive.

The general bulk metric is:

$$ds^2 = e^{-2\sum_{i=-l}^{m} k_i|y-y_i| - 2k_c y - 2c} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \ . \tag{92}$$

And the corresponding 4-dimensional Planck scale is:

$$M_{pl}^2 = M_X^3 \left(T_{-\infty,-l} + T_{m,+\infty} + \sum_{i=-l}^{m-1} T_{i,i+1} \right) \ , \tag{93}$$

where

$$T_{-\infty,-l} = \frac{1}{2\chi_{-\infty}} e^{-2\sigma(y_{-l})} \ , \ T_{m,+\infty} = \frac{1}{2\chi_{+\infty}} e^{-2\sigma(y_m)} \ , \tag{94}$$

if $\chi_{i,i+1} \neq 0$, then

$$T_{i,i+1} = \frac{1}{2\chi_{i,i+1}} \left(e^{-2\sigma(y_{i+1})} - e^{-2\sigma(y_i)} \right) \ , \tag{95}$$

and if $\chi_{i,i+1} = 0$, then

$$T_{i,i+1} = (y_{i+1} - y_i) e^{-2\sigma(y_i)} \ , \tag{96}$$

where

$$\chi_{-\infty} = \sum_{j=-l}^{m} k_j - k_c \ , \ \chi_{+\infty} = \sum_{j=-l}^{m} k_j + k_c \ , \tag{97}$$

$$\chi_{i,i+1} = \sum_{j=i+1}^{m} k_j - \sum_{j=-l}^{i} k_j - k_c \ . \tag{98}$$

By the way, one can easily prove that $T_{i,i+1}$ is positive, which makes sure that the 4-dimensional Planck scale is positive.

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} \ . \tag{99}$$

In this paper, we assume that $M5_{GUT}^{(i)} \equiv M_X$, for $i = -l, ..., m$.

Now, in general, we would like to explain how to solve the gauge hierarchy problem. Using fact that $T_{-\infty,-l} > 0, T_{m,+\infty} > 0, T_{i,i+1} > 0$ where $i = -l, ..., m-1$, and noticing that $\sum_{j=-l}^{m} k_j > |k_c|$, we can prove that if $\sigma(y_j)$ is the minimal value of $\sigma(y_i)$ for all i, then the $j-th$ brane will have positive tension. If $e^{-2(\sigma(y_i)-\sigma(y_j))} << 1$ for all $i \neq j$, then, one obtain:

$$M_{pl}^2 = \frac{M_X^3}{m_c} e^{-2\sigma(y_j)} \ , \tag{100}$$

where m_c is a function of k_c and k_i for all i. And assume the observable brane is at $y = y_{i_o}$ and $M_X = m_c$, we obtain:

$$M_{GUT}^{(i_o)} = M_{pl} e^{-(\sigma(y_{i_o})-\sigma(y_j))} \ . \tag{101}$$

So, if $\sigma(y_{i_o}) - \sigma(y_j) = 34.5$ and 30, we will push the four-dimensional GUT scale in our world to the TeV and 10^5 GeV range, respectively. Of course, the 4-dimensional GUT scale on $j-th$ brane is the highest, but, the $j-th$ brane does not need to have larger brane tension. Furthermore, if $\sigma(y_j) > 0$, and if the $j-th$ brane is the observable brane, we can also solve the gauge hierarchy problem. Assume that $M_{pl} = m_c$, we obtain:

$$M_{GUT}^{(j)} = M_{pl} e^{-\frac{1}{3}\sigma(y_j)} \ , \tag{102}$$

with $\sigma(y_j) = 103.5$ and 90, we can have GUT scale in our world at TeV scale and 10^5 GeV scale, respectively. Obviously, in this case, because of the exponential factor, the five-dimensional Planck scale is very high, 10^{48} GeV and 10^{44} GeV, respectively.

These solutions can be generalized to the solution with Z_2 symmetry. Because of Z_2 symmetry, $k_c = 0$. There are two kinds of such models, one is the odd number of the branes, the other is the even number of the branes. For the first one, one just requires that $k_{-i} = k_i$, $y_{-i} = -y_i$, and $m = l$. For the second case, one just requires that $k_{-i} = k_i$, $y_{-i} = -y_i$, $m = l$, and $k_0 = 0$ (no number 0 brane).

Now, we discuss three examples:

(I) First, we consider one brane with position y_0 and the value of the brane tension divided by $6M_X^3$ is k_0. Therefore, we obtain:

$$\sigma(y) = k_0|y - y_0| + k_c y + c \ . \tag{103}$$

Obviously, we need to require that $k_0 > |k_c|$. One can easily obtain that:

$$\sigma(y_0) = k_c y_0 + c \ . \tag{104}$$

The four-dimensional Planck scale are

$$M_{pl}^2 = M_X^3 \frac{k_0}{k_0^2 - k_c^2} e^{-2k_c y_0 - 2c} \ . \tag{105}$$

Assuming that $M_{pl} = \frac{k_0^2 - k_c^2}{k_0}$, we obtain:

$$M_{GUT}^{(0)} = M_{pl} e^{-\frac{1}{3}k_c y_0 - \frac{1}{3}c} \ , \tag{106}$$

with $k_c y_0 + c = 103.5$ and 90, the GUT scale in our world will be pushed to TeV scale and 10^5 GeV scale range, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively. By the way, if $k_c = 0$, the constant c is the key factor, which is often thought physical irrelevant. The model with $k_c = c = 0$ was discussed in [3] .

In addition, we can obtain the exact relation between M_{pl} and $M_{GUT}^{(0)}$:

$$M_{pl} = M_{GUT}^{(0)} \sqrt{\frac{M_X k_0}{k_0^2 - k_c^2}} \ . \tag{107}$$

In order to solve the gauge hierarchy problem, in general, we might require that M_X is very large compare to k_0 and k_c, or $k_0^2 - k_c^2$ is very smalll.

(II) Two brane cases: their positions are y_0, y_1, respectively, and the values of the brane tension divided by $6M_X^3$ are: k_0, k_1, respectively. Therefore, we obtain:

$$\sigma(y) = k_0|y - y_0| + k_1|y - y_1| + k_c y + c \ , \tag{108}$$

The constrint is $k_0 + k_1 > |k_c|$. One can easily obtain:

$$\sigma(y_0) = k_1(y_1 - y_0) + k_c y_0 + c \ , \tag{109}$$

$$\sigma(y_1) = k_0(y_1 - y_0) + k_c y_1 + c \ . \tag{110}$$

The four-dimensional Planck scale are

$$M_{pl}^2 = M_X^3 \left(\frac{k_0}{k_0^2 - (k_1 - k_c)^2} e^{-2\sigma(y_0)} + \frac{k_1}{k_1^2 - (k_0 + k_c)^2} e^{-2\sigma(y_1)} \right) \ . \tag{111}$$

If $k_0 > 0$, $k_1 > 0$, without loss of generality,assuming that $\sigma(y_0) < \sigma(y_1)$ and $e^{-2(\sigma(y_1) - \sigma(y_0))} << 1$, we obtain

$$M_{pl}^2 = M_X^3 \frac{k_0}{k_0^2 - (k_1 - k_c)^2} e^{-2\sigma(y_0)} \ . \tag{112}$$

If the brane with position y_1 is the observable brane, assuming $M_X = \frac{k_0^2 - (k_1 - k_c)^2}{k_0}$, we obtain:

$$M_{GUT}^{(1)} = M_{pl} e^{-(\sigma(y_1) - \sigma(y_0))} \ . \tag{113}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $(k_0 - k_1 + k_c)(y_1 - y_0) = 34.5$ and 30, respectively. And the value of $\sigma(y_0)$ determines the relation between M_{pl} and M_X. Explicit example: $k_0 = k_1 = k_c > 0$, $k_c(y_1 - y_0) = 34.5$ and 30, which is not discussed in [19]. From this explicit example, we conclude that k_c is also an important factor to solve the gauge hierarchy problem.

If the brane with position y_0 is the observable brane, we can solve the gauge hierarchy problem only when $\sigma(y_0) > 0$. Assuming that $M_{pl} = \frac{k_0^2 - (k_1 - k_c)^2}{k_0}$ and $e^{-2(\sigma(y_1) - \sigma(y_0))} << 1$, we obtain:

$$M_{GUT}^{(0)} = M_{pl} e^{-\frac{1}{3}\sigma(y_0)} \ , \tag{114}$$

with $\sigma(y_0) = 103.5$ and 90, we can have GUT scale in our world at TeV scale and 10^5 GeV scale, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

If $k_0 > 0, k_1 < 0$, the solution to the gauge hierarchy problem is similar to that in just above paragraph.

(III) Three brane case: their positions are y_1, y_2, y_3, respectively, and the values of the brane tension divided by $6M_X^3$ are: k_1, k_2, k_3, respectively. And the constraint is $k_1 + k_2 + k_3 > |k_c|$. Therefore, we obtain:

$$\sigma(y) = \sum_{i=1}^{3} k_i |y - y_i| + k_c y + c \ , \tag{115}$$

$$\sigma(y_1) = k_2(y_2 - y_1) + k_3(y_3 - y_1) + k_c y_1 + c \ , \tag{116}$$

$$\sigma(y_2) = k_1(y_2 - y_1) + k_3(y_3 - y_2) + k_c y_2 + c \ , \tag{117}$$

$$\sigma(y_3) = k_1(y_3 - y_1) + k_2(y_3 - y_2) + k_c y_3 + c \ . \tag{118}$$

The four-dimensional Planck scale are

$$\begin{aligned} M_{pl}^2 &= \frac{M_X^3}{2} \left(\frac{1}{k_1 + k_2 + k_3 - k_c} e^{-2\sigma(y_1)} + \frac{1}{k_2 + k_3 - k_c - k_1} (e^{-2\sigma(y_2)} - e^{-2\sigma(y_1)}) \right. \\ &+ \frac{1}{k_3 - k_c - k_1 - k_2} (e^{-2\sigma(y_3)} - e^{-2\sigma(y_2)}) \\ &+ \left. \frac{1}{k_1 + k_2 + k_3 + k_c} e^{-2\sigma(y_3)} \right) \ . \end{aligned} \tag{119}$$

If there exist at least two brane which have positive tensions. Without loss of generality, assuming the brane with position y_1 has positive tension, $\sigma(y_1) < \sigma(y_2)$, and $\sigma(y_1) < \sigma(y_3)$. If $e^{-2(\sigma(y_2) - \sigma(y_1))} << 1$ and $e^{-2(\sigma(y_3) - \sigma(y_1))} << 1$, we obtain

$$M_{pl}^2 = M_X^3 \frac{k_1}{k_1^2 - (k_2 + k_3 - k_c)^2} e^{-2\sigma(y_1)} \ . \tag{120}$$

If the brane with position y_j where $j = 2, 3$ is the observable brane, and assuming $M_X = \frac{k_1^2 - (k_2 + k_3 - k_c)^2}{k_1}$, we obtain:

$$M_{GUT}^{(j)} = M_{pl} e^{-(\sigma(y_j) - \sigma(y_1))} \ . \tag{121}$$

So, the GUT scale in our world will be at TeV scale and 10^5 GeV scale range if $\sigma(y_j) - \sigma(y_1) = 34.5$ and 30, respectively. And the value of $\sigma(y_1)$ determines the relation between M_{pl} and M_X. For an explicit example, $y_1 = -y_3$, $y_2 = 0$, and $k_1 = k_3 = k_c$, $k_2 = -\frac{1}{2} k_c$.

$$\sigma(y_1) = -\frac{1}{2} k_c y_3 + c \ , \ \sigma(y_2) = 2 k_c y_3 + c \ , \tag{122}$$

$$\sigma(y_3) = \frac{5}{2} k_c y_3 + c \ . \tag{123}$$

If $M_X = \frac{3}{4} k_c$ and the brane with position y_3 is the observable brane, we obtain

$$M_{GUT}^{(3)} = M_{pl} e^{-3k_c y_3} \ . \tag{124}$$

So, one can easily solve the gauge hierarchy problem.

If the brane with position y_1 is the observable brane, we can solve the gauge hierarchy problem only if $\sigma(y_1) > 0$. Assuming that $M_{pl} = \frac{k_1^2 - (k_2+k_3-k_c)^2}{k_1}$ and $e^{-2(\sigma(y_j)-\sigma(y_1))} << 1$ where $j = 2, 3$, we obtain:

$$M_{GUT}^{(1)} = M_{pl} e^{-\frac{1}{3}\sigma(y_1)} \ , \tag{125}$$

if $\sigma(y_1) = 103.5$ and 90, the GUT scale in our world will be about TeV and 10^5 GeV, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

For just one brane with positive tension case, the ressult is similar to that in just above paragraph.

3.2 Space-Time $M^4 \times R^1/Z_2$

Now, we consider the space-time is $M^4 \times R^1/Z_2$. Similar to the section 2, the solution with Z_2 symmetry and odd number of 3-branes in subsection 3.1 can also be generalized to the case in which the fifth dimension is R^1/Z_2. One just introduces the equivalence classes: $y \sim -y$ and $i - th\ brane \sim (-i) - th\ brane$, and then, modules the equivalence classes. Noticing that the brane tension V_0 is half of the original value, i. e., $V_0 = 3k_0 M_X^3$, we obtain the sum of brane tensions is positive, too. In short, the results are similar to that in above subsection except that the range of y is from 0 to $+\infty$. And we will have $m+1$ brane with positions $y_0 = 0 < y_1 < y_2 < ... < y_m < +\infty$.

Assuming that $k_T = \sum_{j=1}^{m} k_j$, we obtain the the general solution to $\sigma(y)$ is:

$$\sigma(y) = \sum_{i=0}^{m} k_i |y - y_i| + k_T y + c \ . \tag{126}$$

And the corresponding 4-dimensional Planck scale is:

$$M_{pl}^2 = M_X^3 \left(T_{m,+\infty} + \sum_{i=0}^{m-1} T_{i,i+1} \right) \ , \tag{127}$$

where

$$T_{m,+\infty} = \frac{1}{2\chi_{+\infty}} e^{-2\sigma(y_m)} \ , \tag{128}$$

if $\chi_{i,i+1} \neq 0$, then

$$T_{i,i+1} = \frac{1}{2\chi_{i,i+1}} \left(e^{-2\sigma(y_{i+1})} - e^{-2\sigma(y_i)} \right) \ , \tag{129}$$

and if $\chi_{i,i+1} = 0$, then

$$T_{i,i+1} = (y_{i+1} - y_i) e^{-2\sigma(y_i)} ~, \tag{130}$$

where

$$\chi_{+\infty} = \sum_{j=0}^{m} k_j + k_T ~, \tag{131}$$

$$\chi_{i,i+1} = \sum_{j=i+1}^{m} k_j - \sum_{j=0}^{i} k_j - k_T ~. \tag{132}$$

By the way, one can easily prove that $T_{i,i+1}$ is positive, which makes sure that the 4-dimensional Planck scale is positive.

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} ~. \tag{133}$$

Now, we discuss the example, for two brane case, it is equivalent to above 3-brane case with Z_2 symmetry, so, we donot discuss it here. We discuss three 3-brane case: their positions are: $y_0 = 0$, y_1, y_2, respectively, and the values of the brane tensions divided by $6M_X^3$ are: $\frac{1}{2}k_0$, k_1, k_2, respectively. Therefore, noticing $k_T = k_1 + k_2$, we obtain:

$$\sigma(y) = \sum_{i=0}^{2} k_i |y - y_i| + (k_1 + k_2) y + c ~, \tag{134}$$

$$\sigma(y_0) = k_1 y_1 + k_2 y_2 + c ~, \tag{135}$$

$$\sigma(y_1) = k_0 y_1 + k_2 (y_2 - y_1) + (k_1 + k_2) y_1 + c ~, \tag{136}$$

$$\sigma(y_2) = k_0 y_2 + k_1 (y_2 - y_1) + (k_1 + k_2) y_2 + c ~. \tag{137}$$

The four-dimensional Planck scale are:

$$\begin{aligned} M_{pl}^2 &= \frac{M_X^3}{2} \left(-\frac{1}{k_0} (e^{-2\sigma(y_1)} - e^{-2\sigma(y_0)}) - \frac{1}{k_0 + 2k_1} (e^{-2\sigma(y_2)} - e^{-2\sigma(y_1)}) \right. \\ &\quad \left. + \frac{1}{k_0 + 2k_1 + 2k_2} e^{-2\sigma(y_2)} \right) ~. \end{aligned} \tag{138}$$

If there exists at least two branes which have positive tensions. Without loss of generality, assuming the brane with position y_0 has positive tension, $\sigma(y_0) < \sigma(y_1)$, and $\sigma(y_0) < \sigma(y_2)$. If $e^{-2(\sigma(y_1) - \sigma(y_0))} << 1$ and $e^{-2(\sigma(y_2) - \sigma(y_0))} << 1$, we obtain

$$M_{pl}^2 = \frac{M_X^3}{2k_0} e^{-2\sigma(y_0)} ~. \tag{139}$$

If the brane with position y_j where $j = 1, 2$ is the observable brane, we can solve the gauge hierarchy problem. Assuming $M_X = 2k_0$, we obtain:

$$M_{GUT}^{(j)} = M_{pl} e^{-(\sigma(y_j) - \sigma(y_0))} ~. \tag{140}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $\sigma(y_j) - \sigma(y_0) = 34.5$ and 30, respectively. And the value of $\sigma(y_0)$ determines the relation between M_{pl} and M_X.

If the brane with position y_0 is the observable brane, the gauge hierarchy problem can be solved only if $\sigma(y_0) > 0$. Assuming that $M_{pl} = 2k_0$ and $e^{-2(\sigma(y_j)-\sigma(y_0))} << 1$ where $j = 1, 2$, we obtain:

$$M^{(0)}_{GUT} = M_{pl} e^{-\frac{1}{3}\sigma(y_0)} \ , \tag{141}$$

with $\sigma(y_0) = 103.5$ and 90, we can have GUT scale in our world at TeV scale and 10^5 GeV scale, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

For just one brane with positive tension case, the discussion is similar to that in just above paragraph.

3.3 Space-Time $M^4 \times S^1$

Now we consider the fifth dimension is S^1, i. e., the space-time is $M^4 \times S^1$. Assuming we have $l + m + 1$ parallel 3-branes, and their fifth coordinastes are: $-\pi\rho \leq y_{-l} < y_{-l+1} < ... < y_{-1} < y_0 < y_1 < ... < y_{m-1} < y_m < \pi\rho$. The Lagrangian, the Einstein equation and the differential equations of $\sigma(y)$ are the same as that in subsection 3.1, except we require that: $|y| \leq \pi\rho$, $\sigma(-\pi\rho) = \sigma(\pi\rho)$, and the defintion of $\Lambda(y)$. The $\Lambda(y)$ is defined as the following:

$$\begin{aligned} \Lambda(y) &= \sum_{i=-l+1}^{m} \Lambda_i \left(\theta(y - y_{i-1}) - \theta(y - y_i)\right) + \Lambda_*(\theta(y - y_m) \\ &\quad +\theta(-y + y_{-l}))\ . \end{aligned} \tag{142}$$

The general solution to the differential equations of $\sigma(y)$ is:

$$\sigma(y) = \sum_{i=-l}^{m} k_i |y - y_i| + k_c y + c \ . \tag{143}$$

The relations between the k_i and V_i, and the relations between the k_i and Λ_i are:

$$V_i = 6 k_i M_X^3 \ , \tag{144}$$

$$\Lambda_i = -6 M_X^3 (\sum_{j=i}^{m} k_j - \sum_{j=-l}^{i-1} k_j - k_c)^2 \ , \tag{145}$$

$$\Lambda_* = -6 M_X^3 k_c^2 \ , \tag{146}$$

$\frac{d\sigma(y)}{dy}$ is a piece-wise continuous function or sectionally continuous function for it has finite jumps because of finite branes, and the jump is $2k_i$ when it pass the $i-th$ brane. In fact, $\frac{d\sigma(y)}{dy}$ is sectional constant. In addition, if starting a point not belong to any brane, we wind around the circle one times, the sum of jump is $2\sum_{i=-l}^{m} k_i$

which is equal to zero for $\frac{d\sigma(y)}{dy}$ is sectional constant. Therefore, we obtain the sum of the brane tensions should be zero, or

$$\sum_{j=-l}^{m} k_j = 0 \ . \tag{147}$$

In addition, from $\sigma(-\pi\rho) = \sigma(\pi\rho)$, we obtain:

$$\sum_{j=-l}^{m} k_j y_j = k_c \pi\rho \ . \tag{148}$$

And the corresponding 4-dimensional Planck scale is:

$$M_{pl}^2 = M_X^3 \left(T_{m,-l} + \sum_{i=-l}^{m-1} T_{i,i+1} \right) \ , \tag{149}$$

where if $k_c \neq 0$, then

$$T_{m,-l} = -\frac{1}{2k_c} \left(e^{-2\sigma(y_{-l})} - e^{-2\sigma(y_m)} \right) \ , \tag{150}$$

and if $k_c = 0$, then

$$T_{m,-l} = (2\pi\rho + y_{-l} - y_m) \, e^{-2\sigma(y_{-l})} \ , \tag{151}$$

if $\chi_{i,i+1} \neq 0$, then

$$T_{i,i+1} = \frac{1}{2\chi_{i,i+1}} \left(e^{-2\sigma(y_{i+1})} - e^{-2\sigma(y_i)} \right) \ , \tag{152}$$

and if $\chi_{i,i+1} = 0$, then

$$T_{i,i+1} = (y_{i+1} - y_i) e^{-2\sigma(y_i)} \ , \tag{153}$$

where

$$\chi_{i,i+1} = \sum_{j=i+1}^{m} k_j - \sum_{j=-l}^{i} k_j - k_c \ . \tag{154}$$

As before, one can easily prove that $T_{i,i+1}$ is positive, which makes sure that the 4-dimensional Planck scale is positive.

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} \ . \tag{155}$$

This solution can be generalized to the solution with Z_2 symmetry. There are four kinds of such models, two kinds of such models have odd number of branes, the other two have even number of branes. For the odd number of branes: (I) one requires that: $k_{-i} = k_i$, $y_{-i} = -y_i$, $m = l$ and $k_c = 0$. (II) one requires that: $k_{-i} = k_i$, $y_{-i} = -y_i$ for $1 \leq i \leq m$, $l = m+1$, $k_0 = 0$ (no number 0 brane), $y_{-l} = -\pi\rho$ and $k_c = -k_{-l}$. For the even number of branes; (I) one requires that $k_{-i} = k_i$, $y_{-i} = -y_i$, $m = l$, $k_0 = 0$ (no number 0 brane) and $k_c = 0$. (II) $k_{-i} = k_i$, $y_{-i} = -y_i$, for $1 \leq i \leq m$, $l = m+1$, $k_0 \neq 0$, $y_{-l} = -\pi\rho$ and $k_c = -k_{-l}$.

Let us discuss two simple models.

(I) Three brane case: their positions are y_1, y_2, y_3, respectively, and the values of the brane tensions divided by $6M_X^3$ are: k_1, $-(k_1+k_3)$, k_3, respectively, where $k_1 > 0$, $k_3 > 0$. Therefore, we obtain:

$$\sigma(y) = k_1|y-y_1| - (k_1+k_3)|y-y_2| + k_3|y-y_3| + k_c y + c \ . \tag{156}$$

The constraint equation is

$$k_1 y_1 - (k_1+k_3) y_2 + k_3 y_3 = k_c \pi\rho \ . \tag{157}$$

and

$$\sigma(y_1) = k_c(\pi\rho + y_1) + c \ , \tag{158}$$

$$\sigma(y_2) = 2k_1(y_2 - y_1) + k_c(\pi\rho + y_2) + c \ , \tag{159}$$

$$\sigma(y_3) = k_c(y_3 - \pi\rho) + c \ . \tag{160}$$

Without loss of generality, we assume $k_c > 0$, and then, obtain $\sigma(y_3) < \sigma(y_1) < \sigma(y_2)$.

The four-dimensional Planck scale are

$$\begin{aligned} M_{pl}^2 &= \frac{M_X^3}{2}\left(-\frac{1}{k_c}(e^{-2\sigma(y_1)} - e^{-2\sigma(y_3)}) - \frac{1}{2k_1+k_c}(e^{-2\sigma(y_2)} - e^{-2\sigma(y_1)}) \right. \\ &\quad \left. + \frac{1}{2k_3 - k_c}(e^{-2\sigma(y_3)} - e^{-2\sigma(y_2)})\right) \ . \end{aligned} \tag{161}$$

If $e^{-2(\sigma(y_1)-\sigma(y_3))} << 1$, we obtain:

$$M_{pl}^2 = M_X^3 \frac{k_3}{(2k_3 - k_c)k_c} e^{-2k_c(y_3 - \pi\rho) - 2c} \ . \tag{162}$$

Because of the constraint equation, $2k_3 - k_c > 0$. If the brane with position y_1 is the observable brane, we can solve the gauge hierarchy problem. Assuming $M_X = \frac{(2k_3-k_c)k_c}{k_3}$, we obtain:

$$M_{GUT}^{(1)} = M_{pl} e^{-k_c(2\pi\rho + y_1 - y_3)} \ . \tag{163}$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $k_c(2\pi\rho + y_1 - y_3) = 34.5$ and 30, respectively. And the value of $\sigma(y_3)$ determines the relation between M_{pl} and M_X.

If the brane with position y_3 is the observable brane, we can solve the gauge hierarchy problem only if $\sigma(y_3) > 0$. Assuming that $M_{pl} = \frac{(2k_3-k_c)k_c}{k_3}$ and $e^{-2(\sigma(y_1)-\sigma(y_3))} <<$ 1, we obtain:

$$M_{GUT}^{(3)} = M_{pl} e^{-\frac{1}{3}\sigma(y_3)} \ , \tag{164}$$

with $\sigma(y_3) = 103.5$ and 90, we can have GUT scale in our world at TeV scale and 10^5 GeV scale, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

(II) Four brane with Z_2 symmetry case. The branes' positions are $y_{-2} = -\pi\rho$, $y_{-1} = -y_1$, $y_0 = 0$, y_1, respectively, and the values of the brane tension divided by $6M_X^3$ are: $2k_2$, $-(k_0+k_2)$, $2k_0$, $-(k_0+k_2)$, respectively, where $k_0 > 0, k_2 > 0$. Therefore, we obtain:

$$\sigma(y) = -(k_2+k_0)(|y-y_1| + |y+y_1|) + 2k_0|y| + c \ , \tag{165}$$

then,

$$\sigma(y_{-2}) = -2k_2\pi\rho + c \ , \ \sigma(y_0) = -2(k_2+k_0)y_1 + c \ , \tag{166}$$

$$\sigma(y_{-1}) = \sigma(y_1) = -2k_2 y_1 + c \ . \tag{167}$$

The four-dimensional Planck scale are

$$\begin{aligned} M_{pl}^2 &= \frac{M_X^3}{4}\left(-\frac{1}{k_2}(e^{-2\sigma(y_{-1})} - e^{-2\sigma(y_{-2})}) + \frac{1}{k_0}(e^{-2\sigma(y_0)} - e^{-2\sigma(y_{-1})}) \right. \\ &\left. -\frac{1}{k_0}(e^{-2\sigma(y_1)} - e^{-2\sigma(y_0)}) + \frac{1}{k_2}(e^{-2\sigma(y_{-2})} - e^{-2\sigma(y_1)})\right) \ . \end{aligned} \tag{168}$$

Without loss of generality, we assume that $k_0 > 0$, and $\sigma(y_{-2}) < \sigma(y_0)$ or $k_2\pi\rho - (k_2 + k_0)y_1 > 0$. If the brane with position y_0 is the observable brane, the gauge hierarchy problem can also be solved. Assuming that $e^{-2(\sigma(y_0)-\sigma(y_{-2}))} << 1$, and $M_X = 2k_2$, one obtains:

$$M_{GUT}^{(0)} = M_{pl} e^{-(\sigma(y_0)-\sigma(y_{-2}))} \ . \tag{169}$$

So, one can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $\sigma(y_0) - \sigma(y_{-2}) = 2(k_2\pi\rho - (k_2+k_0)y_1) = 34.5$ and 30, respectively. And the value of $\sigma(y_{-2})$ determines the relation between M_{pl} and M_X.

3.4 Space-Time $M^4 \times S^1/Z_2$

The solution with Z_2 symmetry in the last subsection can also be generalized to the case in which the fifth dimension is S^1/Z_2. One just requires that $k_{-i} = k_i$, $y_{-i} = -y_i$, for $1 \leq i \leq m$, $l = m+1$, $y_{-l} = -\pi\rho$, and $k_c = -k_{-l}$, then introduces the equivalence classes: $y \sim -y$ and $i-th\ brane \sim (-i)-th\ brane$ for $i = 1, ..., m$. After module the eqivalence classes, we obtain the models on $M^4 \times S^1/Z_2$. We renumber $-l-th$ brane as $(m+1)-th$ brane, so, $y_{m+1} = \pi\rho$. Using splitting brane method, we obtain the brane tension V_0 and V_{m+1} is half of the original value, i. e., $V_0 = 3k_0 M_X^3$, $V_{m+1} = 3k_{m+1}M_X^3$. Of course, the sum of the brane tensions is zero, too. And we will have $m+2$ branes with position $y_0 = 0 < y_1 < ... < y_m < y_{m+1} = \pi\rho$. By the way, $k_0 + k_{m+1} + 2\sum_{i=1}^m k_i = 0$.

The solution of $\sigma(y)$ is:

$$\sigma(y) = k_0|y| + \sum_{i=1}^{m} k_i(|y-y_i| + |y+y_i|) + c \ . \tag{170}$$

And we can rewrite it as

$$\sigma(y) = \sum_{i=1}^{m} k_i |y - y_i| + k_T y + c \ , \tag{171}$$

where $k_T = \frac{1}{2}(k_0 - k_{m+1})$.

And the corresponding 4-dimensional Planck scale is:

$$M_{pl}^2 = M_X^3 \left(\sum_{i=0}^{m} T_{i,i+1} \right) \ , \tag{172}$$

where if $\chi_{i,i+1} \neq 0$, then

$$T_{i,i+1} = \frac{1}{2\chi_{i,i+1}} \left(e^{-2\sigma(y_{i+1})} - e^{-2\sigma(y_i)} \right) \ , \tag{173}$$

and if $\chi_{i,i+1} = 0$, then

$$T_{i,i+1} = (y_{i+1} - y_i) e^{-2\sigma(y_i)} \ , \tag{174}$$

where

$$\chi_{i,i+1} = \sum_{j=i+1}^{m} k_j - \sum_{j=1}^{i} k_j - k_T \ . \tag{175}$$

We use the notation: $i = 0$ and $i = m$, $\sum_{j=1}^{i} k_j = 0$ and $\sum_{j=i+1}^{m} k_j = 0$, respectively. By the way, one can easily prove that $T_{i,i+1}$ is positive, which makes sure that the 4-dimensional Planck scale is positive.

In addition, the four-dimensional GUT scale on i-th brane $M_{GUT}^{(i)}$ is related to the five-dimensional GUT scale on i-th brane $M5_{GUT}^{(i)}$:

$$M_{GUT}^{(i)} = M5_{GUT}^{(i)} e^{-\sigma(y_i)} \ . \tag{176}$$

Now, we discuss the simple example. The three brane case is equivalent to above four 3-brane case with Z_2 symmetry module the equivalence classes. So, we will not discuss three 3-brane case, here. We discuss four 3-brane case: their positions are $y_0 = 0$, y_1, y_2, $y_3 = \pi\rho$, respectively, and the values of the brane tension divided by $6M_X^3$ are: $\frac{1}{2}(-k_1 + k_2 + k_T)$, k_1, $-k_2$, $-\frac{1}{2}(k_1 - k_2 + k_T)$, respectively, where $k_1 > 0, k_2 > 0$. Therefore, we obtain:

$$\sigma(y) = k_T y + k_1 |y - y_1| - k_2 |y - y_2| + c \ , \tag{177}$$

$$\sigma(y_0) = k_1 y_1 - k_2 y_2 + c \ , \tag{178}$$

$$\sigma(y_1) = -k_2 (y_2 - y_1) + k_T y_1 + c \ , \tag{179}$$

$$\sigma(y_2) = k_1 (y_2 - y_1) + k_T y_2 + c \ , \tag{180}$$

$$\sigma(y_3) = (k_1 - k_2 + k_T)\pi\rho - (k_1 y_1 - k_2 y_2) + c \ . \tag{181}$$

$$
\begin{aligned}
M_{pl}^2 &= -\frac{M_X^3}{2}\left(\frac{1}{k_2+k_T-k_1}(e^{-2\sigma(y_1)}-e^{-2\sigma(y_0)})\right.\\
&\quad+\frac{1}{k_1+k_2+k_T}(e^{-2\sigma(y_2)}-e^{-2\sigma(y_1)})\\
&\quad\left.+\frac{1}{k_1-k_2+k_T}(e^{-2\sigma(y_3)}-e^{-2\sigma(y_2)})\right)\ . \qquad (182)
\end{aligned}
$$

Without loss of generality, we assume that $k_T > 0$ and $k_2 + k_T > k_1$. Then, the brane at y_0 has positive tension and $\sigma(y_0) < \sigma(y_1)$. Assuming $e^{-2(\sigma(y_j)-\sigma(y_0))} << 1$ for $j = 1, 2, 3$, and $M_X = 2(k_2 + k_T - k_1)$, if the brane with position y_1 is the observable brane, we can solve the gauge hierarchy problem:

$$
M_{GUT}^{(1)} = M_{pl}e^{-(k_2-k_1+k_T)y_1}\ . \qquad (183)
$$

So, we can push the GUT scale in our world to TeV scale and 10^5 GeV scale range if $(k_2 - k_1 + k_T)y_1 = 34.5$ and 30, respectively. And the value of $\sigma(y_0)$ determines the relation between M_{pl} and M_X.

If the brane with position y_0 is the observable brane, one can solve the gauge hierarchy problem only if $\sigma(y_0) > 0$, i. e., $k_1y_1 > k_2y_2$. Assuming that $M_{pl} = 2(k_2 + k_T - k_1)$ and $e^{-2(k_2-k_1+k_T)y_1} << 1$, one obtains:

$$
M_{GUT}^{(0)} = M_{pl}e^{-\frac{1}{3}\sigma(y_0)}\ , \qquad (184)
$$

with $\sigma(y_0) = 103.5$ and 90, one can push the GUT scale in our world to TeV scale and 10^5 GeV scale range, respectively. The five-dimensional Planck scale is 10^{48} GeV and 10^{44} GeV, respectively.

4 Conclusion

If the fifth dimension is one-dimensional connected manifold, up to diffeomorphic, the only possible space-time will be $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$. And there exist two possibilities on cosmology constant: the cosmology constant is constant along the fifth dimension, and the cosmology constant is sectional constant along the fifth dimension. For any point in $M^4 \times R^1$, $M^4 \times R^1/Z_2$, $M^4 \times S^1$ and $M^4 \times S^1/Z_2$, which is not belong to any brane and the section where the cosmology constant is zero, there is a neighborhood which is diffeomorphic to (or a slice of) AdS_5 space. We construct the general models with parallel 3-branes on those kinds of the space-time and with constant/sectional constant cosmology constant, and point out that for compact fifth dimension, the sum of the brane tensions is zero, for non-compact fifth dimension, the sum of the brane tensions is positive. We assume the observable brane which includes our world should have positive tension, and conclude that in those general scenarios, the 5-dimensional GUT scale on each brane can be indentified as the 5-dimensional Planck scale, but, the 4-dimensional Planck scale is generated from the low 4-dimensional GUT scale exponentially in our world. We also give some simple models to show explicitly how to solve the gauge hierarchy problem.

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