Tests of M-Theory from $\mathcal{N}=2$ Seiberg-Witten Theory [1]

Isabel P. Ennes [2]
Martin Fisher School of Physics
Brandeis University, Waltham, MA 02254

Stephen G. Naculich [3]
Department of Physics
Bowdoin College, Brunswick, ME 04011

Henric Rhedin
Celsius Consultants, Chalmers Teknikpark
S-412 88 Göteborg, Sweden.

Howard J. Schnitzer[4]
Martin Fisher School of Physics
Brandeis University, Waltham, MA 02254

Abstract

Methods are reviewed for computing the instanton expansion of the prepotential for $\mathcal{N}=2$ Seiberg-Witten (SW) theory with *non*-hyperelliptic curves. These results, if compared with the instanton expansion obtained from the microscopic Lagrangian, will provide detailed tests of M-theory. We observe group-theoretic regularities of $\mathcal{F}_{1-\text{inst}}$ which allow us to "reverse engineer" a SW curve for $SU(N)$ gauge theory with two hypermultiplets in the antisymmetric representation and $N_f \leq 3$ hypermultiplets in the fundamental representations, a result not yet available by other methods. Consistency with M-theory requires a curve of infinite order, which we identify as a decompactified version of elliptic models of the type described by Donagi and Witten, Uranga, and others. This leads us to a brief discussion of some elliptic models that relate to our work.

---

1. Objectives

It is our purpose to present a method for obtaining precise tests of M-theory using $\mathcal{N} = 2$ Seiberg-Witten supersymmetric (susy) gauge theory [1]. Although the string community believes in M-theory, it must nevertheless be subjected to detailed verification for the same reasons that one subjects quantum electrodynamics to such precision tests as the measurement and computation of $g - 2$. In our context, M-theory provides SW curves for low-energy effective $\mathcal{N} = 2$ susy gauge theories, which in principle allows one to compute the instanton expansion of the prepotential of the theory. The results of this calculation must be compared with calculations of the instanton contributions to the prepotential from the microscopic Lagrangian. It is this comparison which provides the tests we are concerned with. It should be noted that what we are considering is the ability of M-theory to make detailed non-perturbative predictions for field theory, as we consider the limit in which gravity has decoupled.

M-theory provides SW curves for effective $\mathcal{N} = 2$ susy gauge theories with hypermultiplets in the both fundamental representation [2, 3] and in higher representations [4]. Since the (hyperelliptic) curves from the former were initially obtained from purely field-theoretic considerations [5], we regard these as *post*-dictions of M-theory, though the agreement is gratifying. In order to obtain genuine tests of M-theory we need to consider situations for which it is not known how to obtain SW curves from field-theoretic arguments alone. Examples of this kind are, for example, $\mathcal{N} = 2$ SU(N) gauge theory with a hypermultiplet in the symmetric or antisymmetric representation [4]. In such cases M-theory gives the only known predictions of the relevant SW curves, which happen to be *non*-hyperelliptic curves. If one can extract the instanton expansion for these examples, and compare these to results from a microscopic calculation, one will have genuine tests of M-theory. The problem we faced is that there were no known methods to obtain the instanton expansion. Our solution to this issue will be one of the main themes of this review[6]–[10].

1

One of the intriguing aspects of SW theory is the connection to integrable models: elliptic models in particular [11]–[14]. M-theory provides one method of constructing the spectral curves of elliptic models. Another approach to these problems is that of geometric engineering [15], which we will not discuss here. We will see that our efforts in understanding SW theory with non-hyperelliptic curves leads us in a natural way to the consideration of M-theory and elliptic models. Some aspects of this connection will be the second main theme of this paper.

A schematic chart of some of these connections is shown in Figure 1.
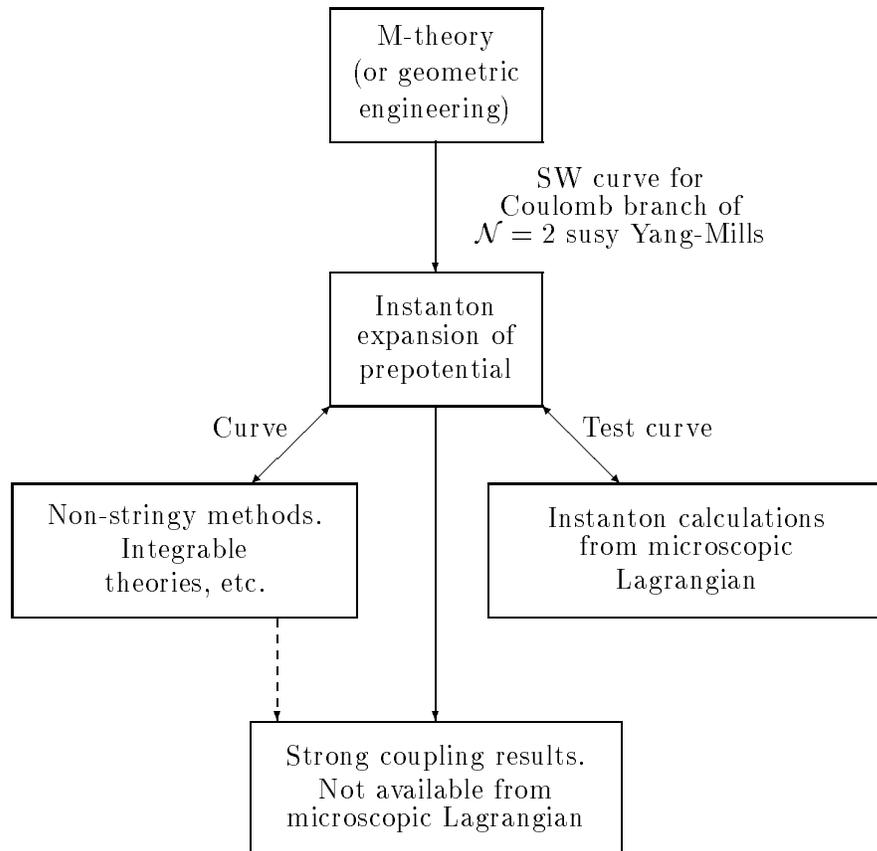
Figure 1

2. Topics considered

1) *Brief introduction* to

    a) $\mathcal{N} = 2$ Seiberg-Witten theory.

    b) M-theory construction of SW curves (Riemann surfaces).

2) a) Instanton expansion for *non*-hyperelliptic curves.

    b) Tests + predictions of M-theory.

3) Reverse engineer a curve for $\mathcal{N} = 2$ SU(N) susy gauge theory, with two antisymmetric representations and $N_f \leq 3$ hypermultiplets.

Require consistency with M-theory which implies a curve of infinite order.

4) Relation to:

    a) Elliptic models.

    b) Integrable models.

5) Some unsolved problems and concluding remarks.

3. Seiberg-Witten Theory

We will be concerned with $\mathcal{N} = 2$ susy Yang-Mills theory in $D = 4$ dimensions, with gauge group $\mathcal{G}$, together with hypermultiplets in some representation R. This theory can be described by a microscopic Lagrangian

$$\mathcal{L}_{\mathrm{micro}} = \frac{1}{4g^2}\, F^a_{\mu\nu}\, F^{\mu\nu a} \; + \; \frac{\theta}{32\pi^2}\, F^a_{\mu\nu}\, \tilde{F}^{\mu\nu a} \; + \; D_\mu \phi^+ D^\mu \phi \; + \; \mathrm{tr}[\phi, \phi^+]^2$$

$$+\, \mathrm{fermion} \; + \; \mathrm{hypermultiplet} \; \mathrm{terms}, \qquad (3.1)$$

with $\mu, \nu = 1$ to 4 and $a = 1$ to dim $\mathcal{G}$. The field strength $F_{\mu\nu}$ and the scalar field ϕ belong to the adjoint representation, as they are the bosonic components of the $\mathcal{N} = 2$ gauge multiplet. The vacuum is described by the condition

$$[\phi, \phi^+] \; = \; 0, \qquad (3.2)$$

which implies $\phi^a = $ constant. One may rotate ϕ^a to the Cartan subalgebra, in which case

$$\text{diag}\,(\phi) \,=\, (a_1, a_2, \ldots)\,, \quad \text{with} \quad \sum_i a_i = 0. \tag{3.3}$$

If all the a_i are distinct, this generically breaks $\mathcal{G}$ to $\text{U}(1)^{rank\ \mathcal{G}}$.

    1) If only ϕ acquires a vacuum expectation value (vev), we define this as the Coulomb branch.

    2) If only the scalar fields in the matter hypermultiplets have a vev, this is the Higgs branch.

    3) There are also mixed branches.

We will focus on the Coulomb branch of these theories.

The breakthrough of Seiberg and Witten [1] was their formulation of the exact solution of low-energy $\mathcal{N} = 2$ susy gauge theories in terms of an effective (Wilsonian) action accurate to two derivatives of the fields. In $D = 4$, the SW program is described in terms of

$$\mathcal{L}_{eff} \,=\, \frac{1}{4\pi}\text{Im} \left(\int \mathrm{d}^4\theta \frac{\partial \mathcal{F}(A)}{\partial A_i} \bar{A}_i + \frac{1}{2} \int \mathrm{d}^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A_i\, \partial A_j} W_i^\alpha W_{\alpha, j} \right) + \text{ higher derivatives}, \tag{3.4}$$

where A^i are $\mathcal{N} = 1$ chiral superfields ($i = 1$ to rank $\mathcal{G}$), $\mathcal{F}(A)$ is the holomorphic prepotential, and W^i is the gauge field strength. In components the effective action is

$$\mathcal{L}_{eff} \,=\, \frac{1}{4}\text{Im}(\tau_{ij})\, F_{\mu\nu}^i F^{\mu\nu j} \,+\, \frac{1}{4}\text{Re}(\tau_{ij})\, F_{\mu\nu}^i \tilde{F}^{\mu\nu\, j}$$

$$+\partial_\mu (a^+)^j \partial^\mu (a_D)_j \,+\, \text{fermions}, \tag{3.5}$$

where in (3.5) we have only exhibited bosonic components of the $\mathcal{N} = 2$ gauge superfield. We define the order parameters a_i as in (3.3), $(a_D)_j = \frac{\partial \mathcal{F}(a)}{\partial a_j}$ denote the dual order parameters, and

$$\tau_{ij} \,=\, \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j}, \tag{3.6}$$

is the coupling or period matrix. Note that $\text{Im}(\tau_{ij}) \geq 0$ for positive kinetic energies.

The holomorphic prepotential can be expressed in terms of a perturbative piece and infinite series of instanton contributions as

$$\mathcal{F}(A) \,=\, \mathcal{F}_{\text{classical}}(A) \,+\, \mathcal{F}_{1-\text{loop}}(A) \,+\, \sum_{d=1}^{\infty} \Lambda^{(2N-I(R))d} \mathcal{F}_{\text{d}-\text{inst}}(A), \tag{3.7}$$

4

where we have specialized the instanton terms to $SU(N)$, since we will concentrate on results for that group. Note that due to a non-renormalization theorem, the perturbative expansion for (3.7) terminates at 1-loop, though there is an infinite series of non-perturbative instanton contributions. In (3.7), Λ is the quantum scale (Wilson cutoff) and $I(R)$ is the Dynkin index of matter hypermultiplet(s) of representation R. Further

$$
\mathcal{F}_{1-\mathrm{loop}}(a) = \frac{i}{4\pi} \sum_{\alpha \in \Delta_+} (a \cdot \alpha)^2 \log \left(\frac{a \cdot \alpha}{\Lambda} \right)^2
$$

$$
-\frac{i}{8\pi} \sum_{w \in W_G} \sum_{j=1}^{N_f} (a \cdot w + m_j)^2 \log \left(\frac{a \cdot w + m_j}{\Lambda} \right)^2 , \tag{3.8}
$$

where α ranges over the positive roots Δ_+ of $\mathcal{G}$, w runs over the weight vectors for a hypermultiplet in the representation R, with mass m, and $a_i = \mathrm{diag}(\phi)$ belongs to the Cartan subalgebra of $\mathcal{G}$. Notice that from perturbation theory $\tau_{ij}(a) \sim \log(a_i - a_j) + \dots$ at large a, which is not single-valued.

The Seiberg-Witten data which (in principle) allow one to reconstruct the prepotential are:

1) A suitable Riemann surface or algebraic curve, dependent on moduli u_i, or equivalently on the order parameters a_i.

2) A preferred meromorphic 1-form $\lambda \equiv$ SW differential.

3) A canonical basis of homology cycles on the surface (A_k, B_k).

4) Computation of period integrals

$$
2\pi i a_k = \oint_{A_k} \lambda, \qquad 2\pi i a_{D,k} = \oint_{B_k} \lambda, \tag{3.9}
$$

where recall $a_{D,k} = \frac{\partial \mathcal{F}(a)}{\partial a_k}$ is the dual order parameter. The program is:

i) find the Riemann surface or algebraic curve appropriate to the given matter content,

ii) compute the period integrals, and

iii) integrate these to find $\mathcal{F}(a)$.

What is known about the required Riemann surfaces? For classical groups, with gauge multiplet and N_f hypermultiplets in the fundamental representation, where N_f is restricted by

the requirement of asymptotic freedom, the curves encountered are *all* hyperelliptic. That is, they are all of the form [5]

$$y^2 + 2A(x)\,y + B(x) = 0, \tag{3.10}$$

where the coefficient functions depend on the moduli, or order parameters. It is important to note that all curves in this class can be found from field-theoretic considerations, and do not require M-theory for their derivation. On the other hand, for $SU(N)$ with matter content

a) one antisymmetric representation and $N_f \leq N + 2$ fundamental representations, or

b) one symmetric representation and $N_f \leq N - 2$ fundamental representations,

the appropriate curves are *not* hyperelliptic, but are cubic, of the form [4]:

$$y^3 + 2A(x)\,y^2 + B(x)\,y + \epsilon(x) = 0. \tag{3.11}$$

It should be emphasized that (3.11) has only been obtained by M-theory. The curve has *not* been obtained by other methods. Therefore, any predictions of $\mathcal{L}_{eff}$, using (3.11), when compared with the analogous predictions of $\mathcal{L}_{\mathrm{micro}}$ should be considered genuine tests of M-theory. The extraction of instanton predictions from curves of the form (3.11) will be the concern of the first-half of these lectures.

In the second-half we will discuss $SU(N)$ gauge theory with two antisymmetric representations and $N_f \leq 3$ for which *no* curve has yet been derived by *any* methods. We will "reverse engineer" a curve, using observed group-theoretic regularities of $\mathcal{F}_{1-\mathrm{inst}}$, and demand consistency with M-theory. This will force us to consider a curve of infinite order. In so doing, we will obtain a decompactified version of an elliptic model of the type of Donagi and Witten, Uranga, and others [11]–[18]. This will lead to a brief discussion of elliptic models, and integrable models, as they relate to our work.

The main task in extracting instanton predictions from curves such as (3.11) is the computation of the period integrals (3.9), and the integration of $\partial \mathcal{F}(a)/\partial a_k$ to obtain $\mathcal{F}(a)$. There are

two principal (complementary) methods to evaluate the period integrals for *hyperelliptic* curves.
These are:

1) Picard-Fuchs differential equations for the period integrals [19]. This gives global information
throughout moduli space, but the complexity of the equations increases rapidly with rank $\mathcal{G}$.

2) Direct evaluation of the period integrals by asymptotic expansion [20, 21, 22]. This method
is not limited by rank $\mathcal{G}$, and gives results in "natural" variables. One can only easily obtain
a few explicit terms of the instanton expansion. However, there exists a nice recursion formula
which can generate the instanton expansion recursively from $\mathcal{F}_{1-\text{inst}}$ [23]. (There are also other
methods, involving WDVV equations [24] or Whitham hierarchies [25], which we do not consider
here.)

The problem we face is how to evaluate period integrals

$$\oint \lambda = \oint \frac{x\,dy}{y}, \tag{3.12}$$

for non-hyperelliptic curves such as (3.11). For hyperelliptic curves, y is given as a square-root,
found from the quadratic curve (3.10), and one can evaluate the resulting integral (3.12) by
asymptotic expansion, for example [20, 21, 22]. For the cubic curve, the exact solution is too
complicated to be useable, while for curves of higher order, even exact solutions are not possible.
Numerical solutions are of no interest, as we want to study the analytic behavior of $\mathcal{F}(a)$ on
the order parameters. In Sec. 5 we present a systematic method for extracting the instanton
expansion for curves such as (3.11), which we argued is necessary if we are to test M-theory
predictions for SW theory. First, we review in the next section how M-theory provides Riemann
surfaces for the SW problem.

4. M-theory and the Riemann Surface

The seminal work on this subject is by Witten [3], who considers IIA string theory lifted to
M-theory. We summarize his discussion. It will be frequently convenient to use the language of
IIA theory in describing the brane structure.

For our first example consider $SU(N_1) \times SU(N_2)$, described by IIA theory on $\mathbb{R}^{10}$. The brane structure for this is

v

$(x_4 + ix_5)$

x_6

① ② ③
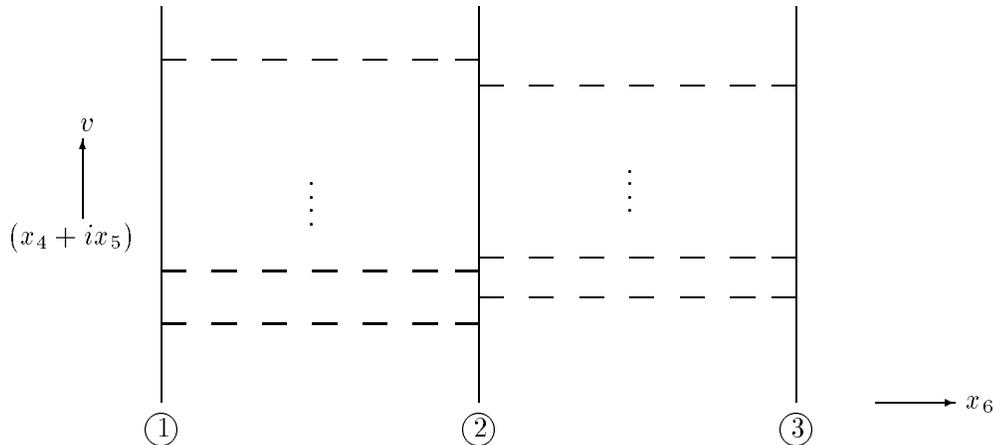
Figure 2

The solid lines represent NS 5-branes and the dashed lines D4-branes suspended between the NS 5-branes, with N_1 of these between ① and ②, and N_2 between ② and ③. The NS 5-branes are located at $x_7 = x_8 = x_9 = 0$, with world-volume x_0 to x_5. Classically, the NS 5-branes are at fixed values of x_6. The D4-branes have world-volume x_0, x_1, x_2, x_3, x_6 with the ends of the D4-branes (classically) at fixed values of x_6. Since the D4-branes are finite in extent in the x_6 direction, the macroscopic world-volume of the D4-branes is (x_0, x_1, x_2, x_3), *i.e.* $d = 4$. We consider the gauge theory on D4-branes.

The NS 5-branes are at definite positions in x_6 only classically. Intersection with the D4-branes creates a disturbance of the NS 5-branes, so that the x_6 position of a NS 5-brane should be measured at $v \sim \infty$, far from all disturbances. For large v,

$$\nabla^2 x_6(v, \bar{v}) = 0, \tag{4.1}$$

which implies

$$x_6 = k \log|v| + \text{const.}, \tag{4.2}$$

for a single D4-brane. For several D4-branes intersecting a given NS 5-brane from the left and

right

$$x_6 = k \sum_{i=1}^{q_L} \log|v - a_i| - k \sum_{i=1}^{q_R} \log|v - b_i| + \text{const.} \qquad (4.3)$$

When there are N parallel D4-branes suspended between a pair of NS 5-branes, generically $SU(N)$ is broken to $U(1)^{N-1}$. One can identify the logarithmic behavior in (4.3) with the large v behavior of the αth gauge theory $SU(N_\alpha)$, with

$$\frac{1}{g_\alpha^2(v)} \sim \frac{x_6^\alpha(v) - x_6^{(\alpha-1)}(v)}{\lambda_{\text{IIA}}} \sim \log v, \qquad (4.4)$$

where λ_{IIA} is the IIA string coupling.

The IIA string theory can be lifted to M-theory, as suggested in Figure 3:
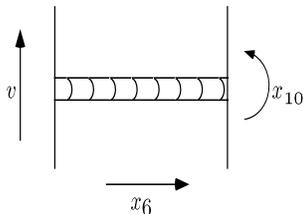
Figure 3

Define

$$v = x_4 + ix_5\,;$$
$$s = \frac{1}{R}(x_6 + ix_{10}), \qquad (4.5)$$

where R is the radius of the 11th dimension (x_{10}), that is

$$x_{10} \to x_{10} + 2\pi R \qquad (\text{periodic}). \qquad (4.6)$$

One then defines

$$\tau_\alpha(v) = \frac{4\pi i}{g_\alpha^2} + \frac{\theta_\alpha}{2\pi}. \qquad (4.7)$$

9

Witten shows that

$$-i\,\tau_\alpha(v) \sim (2k_\alpha - k_{\alpha-1} - k_{\alpha+1})\log v = -(b_0)_\alpha \log v, \tag{4.8}$$

where $-(b_0)_\alpha$ is the coefficient of the 1-loop beta-function for the αth gauge group.

The brane picture lifted to M-theory is closely associated to the SW Riemann surface, since one can consider the type IIA D4-brane in M-theory as an M5-brane wrapped on the S^1 of (4.6). In fact, the type IIA setup with D4-branes and NS 5-branes can be considered in M-theory as a *single* M5-brane with a very complicated (6-dimensional) world-volume $\mathbb{R}^{3,1} \times \Sigma$. One can then identify the brane picture for Σ with the Riemann surface described by the SW curve. In this sense, the Riemann surface acquires a "reality" in M-theory, and is no longer just an auxiliary construct.

Figure 2 describes an $\mathcal{N} = 2$ $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ gauge theory with matter hypermultiplets in the bifundamental representation $(N_1, \bar{N}_2) \oplus (\bar{N}_1, N_2)$. It is also possible to add hypermultiplets in the fundamental representation. To illustrate this, we consider $\mathcal{N} = 2$ $\mathrm{SU}(N)$ gauge theory with $N_f < 2N$.
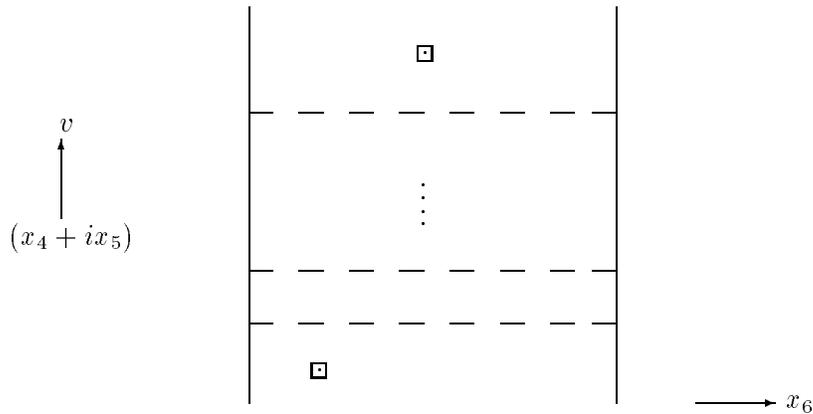
Figure 4

In Fig. 4, we have N D4-branes suspended between two NS 5-branes. In addition, we have N_f hypermultiplets in the fundamental representation denoted by ▫ (in the figure, $N_f = 2$),

which describe D6-branes in $(x_0, x_1, x_2, x_3, x_7, x_8, x_9)$. The bare masses of the hypermultiplets are given by the positions of the D6-branes in v. Define

$$t = \exp(-s) = \exp\left[-(x_6 + ix_{10})/R\right]. \tag{4.9}$$

The curve which describes the positions of the NS 5-branes is

$$t^2 = \frac{1}{4}B(v)^2 - c\prod_{j=1}^{N_f}(v - M_j),$$

$$B(v) = d\prod_{j=1}^{N}(v - e_j), \tag{4.10}$$

with c and d being constants. This curve is hyperelliptic. It agrees with the curve described earlier from field theory [5], so it is not an independent prediction of M-theory.

Another ingredient that we will need is that of orientifold planes. In particular, we will encounter $O6^-$ and $O6^+$ orientifold planes. An $O6^\pm$ orientifold is a 6-plane that extends along the world-volume $(x_0, x_1, x_2, x_3, x_7, x_8, x_9)$ which produces a spacetime reflection. For example, if the $O6^\pm$ is located at $x_4 = x_5 = x_6 = 0$, then these are fixed points of spacetime under $(x_4, x_5, x_6) \rightarrow (-x_4, -x_5, -x_6)$. In other words,

$$(v, x_6) \rightarrow (-v, -x_6). \tag{4.11}$$

(The orientifold also involves a world-sheet parity operation Ω, and $(-1)^{F_L}$ which changes the sign of all left Ramond states. These considerations play no role in our discussion, but emphasize the perturbative nature of the orientifold). The $O6^-$ and $O6^+$ carry RR charge -4 and $+4$ respectively.

We have now all the elements to describe examples of $\mathcal{N} = 2$ theories with non-hyperelliptic SW curves. Consider $SU(N)$ gauge theory with either an antisymmetric or symmetric matter hypermultiplet [4]. The M-theory picture is
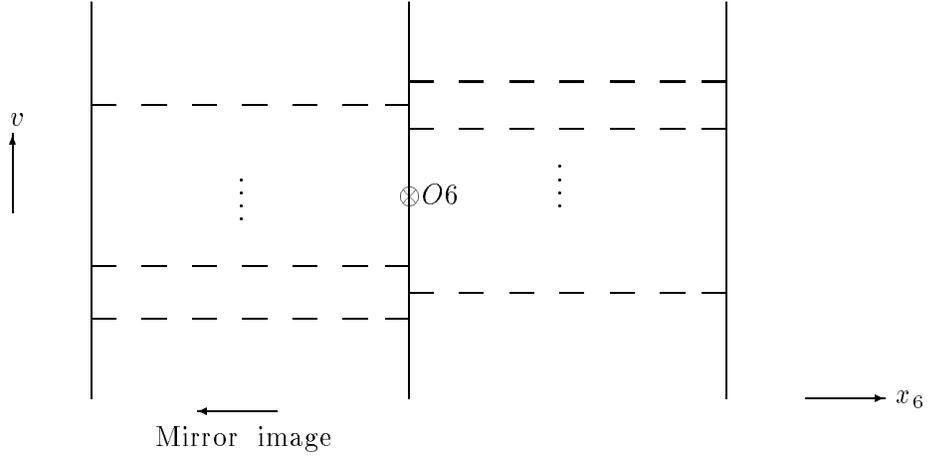
Figure 5

There are 3 parallel NS 5-branes with N D4-branes suspended between each. There is also an $O6$-plane on the central NS 5-brane, which therefore enforces the mirror symmetry (4.11) on the picture. In the absence of the orientifold, one would have $\mathrm{SU}(N) \times \mathrm{SU}(N)$ with matter in the $(N, \bar{N}) \oplus (\bar{N}, N)$ representation. The orientifold "identifies" the two $\mathrm{SU}(N)$ factors, projecting to the diagonal subgroup, giving one hypermultiplet in the antisymmetric representation for $O6^-$, or one hypermultiplet in the symmetric representation for $O6^+$. The curves for these situations have been worked out by Landsteiner and Lopez [4]. It is important to note that the orientifold induces a $\mathbb{Z}_2$ involution in the curve. The curves for these cases are shown in Figure 6.
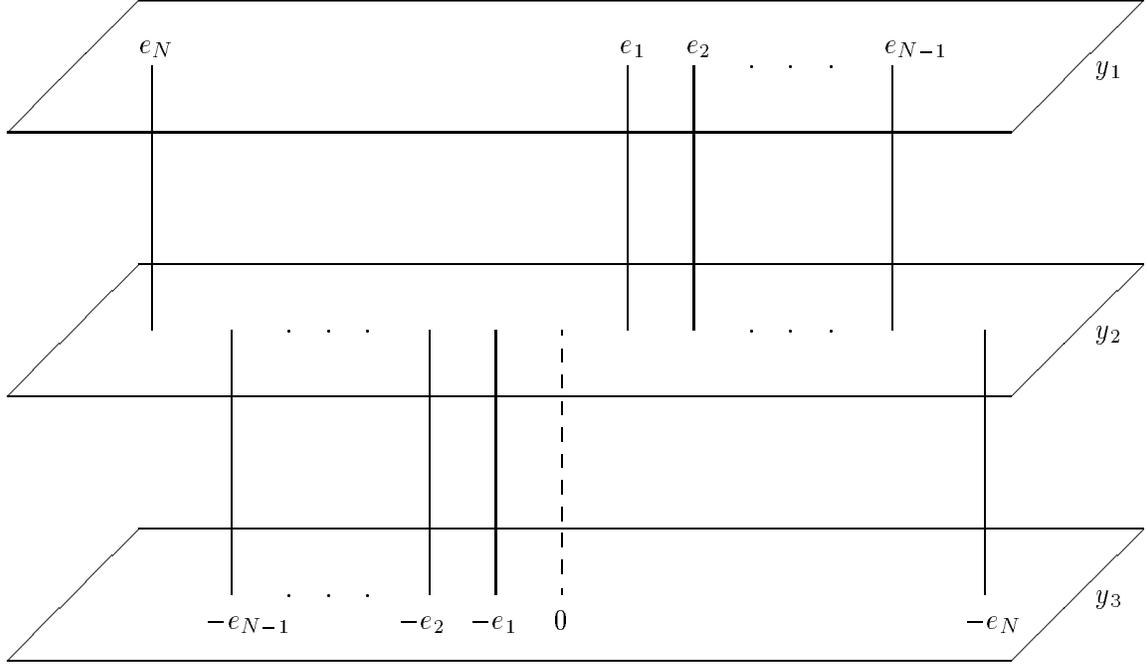
Figure 6 : The sheet structure for the cubic curve corresponding to SU(N) with either one symmetric or one antisymmetric representation. For one symmetric representation, the curve is

$$y^3 + f(x)y^2 + x^2 f(-x)L^2 y + x^6 L^6 = 0,$$

$$f(x) = \prod_{i=1}^{N}(x - e_i), \qquad L^2 = \Lambda^{N-2},$$

$$\text{Involution}: \ y \to \frac{L^4 x^4}{y}, \qquad x \to -x.$$

For one antisymmetric representation, the curve is

$$y^3 + 2A(x)y^2 + B(x)y + L^6 = 0,$$

$$2A(x) = \left[f(x) + 3L^2\right], \qquad B(x) = L^2\left[f(-x) + 3L^2\right],$$

$$f(x) = x^2 \prod_{i=1}^{N}(x - e_i), \qquad L^2 = \Lambda^{N+2},$$

$$\text{Involution}: \ y \to L^4/y, \qquad x \to -x.$$

In Fig. 6 we have relabelled the variables $(t, v) \to (y, x)$ for convenience. Figure 6 can be visualized as type IIA or M-theory pictures by rotating so that y is horizontal and x is vertical. The Riemann surface is a three-fold branched covering of the Riemann sphere, with N square-root branch-cuts connecting Riemann sheets y_1 and y_2, and also y_2 with y_3, with the central values of branch-cuts given by $e_1, \cdots, e_N$, with $\sum_{i=1}^{N} e_i = 0$. Notice that the (classical) positions of the D4-branes correspond to the central values of the branch-cuts. Finally, the curve is cubic, because one needs to describe the y positions of the three NS 5-branes. The analytic problem at hand is how to compute the period integrals for such non-hyperelliptic curves. We describe our method for dealing with this in the next section.

5. Hyperelliptic Perturbation Theory

We have developed a systematic scheme for the instanton expansion for prepotentials associated to non-hyperelliptic curves [6]–[10]. The method will be illustrated for the case of $\mathrm{SU}(N)$ gauge theory with one antisymmetric representation [6] (see Fig. 6). The curve is

$$y^3 + 2A(x)y^2 + B(x)y + \epsilon(x) = 0, \tag{5.1}$$

where $L^2 = \Lambda^{N+2}$, with Λ the quantum scale of the theory,

$$\epsilon = L^6 \; ; \qquad 2A(x) \;\; = \;\; \left[f(x) + 3\, L^2 \right] ,$$

$$f(x) = x^2 \prod_{i=1}^{N} (x - e_i) \; ; \qquad B(x) \;\; = \;\; L^2 \left[f(-x) + 3\, L^2 \right] . \tag{5.2}$$

It is fruitful to regard the last term L^6 in (5.1) as a perturbation. The intuition is that this involves a much higher power of the quantum scale in (5.1) than the other terms, and geometrically it means separating the right-most 5-brane far from the remaining two NS 5-branes. Since the instanton expansion is an expansion in the quantum scale Λ, this idea has a chance of success.

To zeroth approximation we consider (5.1) with $\epsilon = 0$, which is then a hyperelliptic curve, and can be analyzed by previous available methods. This approximation gives $\mathcal{F}_{1-\mathrm{loop}}$ correctly,

but it is not adequate for $\mathcal{F}_{1-\text{inst}}$, so one needs to go beyond the hyperelliptic approximation. Therefore, we develop a systematic expansion in ϵ, which we caution is not the same as an expansion in L, as the coefficient functions $A(x)$ and $B(x)$ depend on L. To make the perturbation expansion look more symmetric define

$$w = y + \frac{2}{3}A \ , \tag{5.3}$$

so that (5.1) is recast as

$$w^3 + \left(B - \tfrac{4}{3}A^2\right)w + \left(\tfrac{16}{27}A^3 - \tfrac{2}{3}AB + \epsilon\right) = 0. \tag{5.4}$$

The solutions to this equation satisfy

$$
\begin{aligned}
(w - w_1)(w - w_2)(w - w_3) &= 0, \\
w_1 + w_2 + w_3 &= 0, \\
w_1 w_2 + w_1 w_3 + w_2 w_3 &= B - \tfrac{4}{3}A^2, \\
w_1 w_2 w_3 &= -\tfrac{16}{27}A^3 + \tfrac{2}{3}AB - \epsilon.
\end{aligned} \tag{5.5}
$$

The zeroth approximation ($\epsilon = 0$) is

$$
\begin{aligned}
\bar{w}_1 &= -\frac{1}{3}A - r, \\
\bar{w}_2 &= -\frac{1}{3}A + r, \\
\bar{w}_3 &= \frac{2}{3}A, \\
r &= \sqrt{A^2 - B}.
\end{aligned} \tag{5.6}
$$

The perturbation expansion for the solution to (5.5) is

$$
\begin{aligned}
w_i = \bar{w}_i + \delta w_i &= \bar{w}_i + \alpha_i \epsilon + \beta_i \epsilon^2 + \cdots \\
\delta(w_1 + w_2 + w_3) &= 0,
\end{aligned}
$$

$$\delta\left(w_1 w_2 + w_2 w_3 + w_3 w_1\right) \quad = \quad 0,$$

$$\delta\left(w_1 w_2 w_3\right) \quad = \quad -\epsilon, \tag{5.7}$$

where to first order

$$\alpha_1 + \alpha_2 + \alpha_3 \quad = \quad 0,$$

$$\bar{w}_1 \alpha_1 + \bar{w}_2 \alpha_2 + \bar{w}_3 \alpha_3 \quad = \quad 0,$$

$$\bar{w}_2 \bar{w}_3 \alpha_1 + \bar{w}_1 \bar{w}_3 \alpha_2 + \bar{w}_1 \bar{w}_2 \alpha_3 \quad = \quad -1, \tag{5.8}$$

with solutions

$$\alpha_1 \quad = \quad \frac{1}{\left(\bar{w}_1 - \bar{w}_2\right)\left(\bar{w}_3 - \bar{w}_1\right)} = \quad - \frac{1}{2r\left(A + r\right)} = \quad - \frac{A - r}{2Br},$$

$$\alpha_2 \quad = \quad \frac{1}{\left(\bar{w}_2 - \bar{w}_3\right)\left(\bar{w}_1 - \bar{w}_2\right)} = \quad \frac{1}{2r\left(A - r\right)} = \quad \frac{A + r}{2Br},$$

$$\alpha_3 \quad = \quad \frac{1}{\left(\bar{w}_3 - \bar{w}_1\right)\left(\bar{w}_2 - \bar{w}_3\right)} = -\frac{1}{B}. \tag{5.9}$$

One can go to the next order, where for example

$$y_3 = -\epsilon \frac{1}{B} - \epsilon^2 \frac{2A}{B^3} + \mathcal{O}(\epsilon^3). \tag{5.10}$$

It is to be observed in (5.6) and (5.9) that sheets y_1 and y_2 are connected by square-root branch-cuts, as indicated by the factors of r, while (5.6), (5.9) and (5.10) show that sheet y_3 is decoupled from y_1 and y_2. In fact this is true in any finite order of the ϵ perturbation theory. However, we know that y_2 and y_3 are also connected by square-root branch-cuts (see Fig. 6). This information is not really lost in our scheme. First note the expansion

$$\frac{1}{\sqrt{B^2 - A}} = \frac{1}{B}\left[1 + \frac{A}{2B^2} + \cdots\right], \tag{5.11}$$

is the structure observed in (5.10), which is the structure of the square root on sheet 3. More importantly, the involution permutes the Riemann sheets, so that if $y_i(x)$ is a solution of (5.1),

16

then

$$\bar{y}_i(x) = \frac{L^4}{y_i(-x)}, \tag{5.12}$$

is also a solution, with the property

$$\begin{aligned}
\bar{y}_1(x) &= y_3(x), \\
\bar{y}_2(x) &= y_2(x), \\
\bar{y}_3(x) &= y_1(x),
\end{aligned} \tag{5.13}$$

so that (5.7)-(5.10), together with the involution captures all the analytic structure of the solution.

The SW differential appropriate to (5.1) is

$$\lambda = x\frac{dy}{y}. \tag{5.14}$$

Since sheet 3 is disconnected in any finite order of our perturbation expansion, we need only consider y_1 and y_2. Let us label the SW differential for these two sheets λ_1 and λ_2 respectively, with

$$\lambda_1 = x\frac{dy_1}{y_1}, \tag{5.15}$$

and λ_2 obtained from (5.15) by $r \rightarrow -r$. The expansion (5.7) induces a comparable expansion for λ, whereby

$$\lambda_1 = (\lambda_1)_I + (\lambda_1)_{II} + \cdots \tag{5.16}$$

with

$$\begin{aligned}
\lambda_I &= \frac{x\left(\frac{A'}{A} - \frac{B'}{2B}\right)}{\sqrt{1 - \frac{B}{A^2}}}\, dx, \\
\lambda_{II} &= -\frac{L^6\left(A - \frac{B}{2A}\right)}{B^2\sqrt{1 - \frac{B}{A^2}}}\, dx,
\end{aligned} \tag{5.17}$$

17

up to terms that do not contribute to period integrals. We note that λ_I is the SW differential obtained from the hyperelliptic approximation ($\epsilon = 0$) to (5.1), and completely determines $\mathcal{F}_{1-\text{loop}}$, while $\lambda_{II} \sim \mathcal{O}(L^2)$, so is of 1-instanton order. Further $(\lambda_1)_{III}, \cdots$, contribute only to 2-instanton order and higher, so we may stop at (5.16)-(5.17) to 1-instanton order.

In order to express the solutions to our problem with economical notation, we define certain "*residue functions*", $R_k(x)$, $S(x)$, $S_0(x)$, and $S_k(x)$, where

$$\frac{R_k(x)}{(x - e_k)} = \frac{3}{f(x)} = \frac{3}{x^2 \prod_{i=1}^{N}(x - e_i)}, \tag{5.18}$$
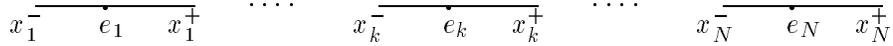
and

$$S(x) = \frac{4f(-x)}{f^2(x)} = \frac{S_0(x)}{x^2} = \frac{S_k(x)}{(x - e_k)^2} = \frac{4(-1)^N \prod_{i=1}^{N}(x + e_i)}{x^2 \prod_{i=1}^{N}(x - e_i)^2}, \tag{5.19}$$

so that

$$S_k(x) = \frac{4(-1)^N \prod_{i=1}^{N}(x + e_i)}{x^2 \prod_{i \neq k}(x - e_i)^2}. \tag{5.20}$$

The functions $S(x)$ and $S_k(x)$ will play a crucial role for understanding of the general features of the instanton expansion of SW problems.

In order to calculate the period integrals, we must first locate the branch-cuts between sheets y_1 and y_2. This is shown in Fig. 7

Figure 7

where the bare order parameters $\{e_i\}$ satisfy

$$\sum_i e_i = 0 \tag{5.21}$$

for a massless hypermultiplet. Branch-cuts occur when $y_1 = y_2$, so that

$$0 = A^2(x_k^{\pm}) - B(x_k^{\pm}) + \frac{L^6 A(x_k^{\pm})}{2B(x_k^{\pm})} + \ldots. \tag{5.22}$$

18

For small L, one may expand (5.22) in powers of L, with the result

$$x_k^\pm = e_k \pm L(S_k(e_k))^{1/2} + L^2 \left[\frac{1}{2}\frac{\partial S_k}{\partial x}(e_k) - R_k(e_k)\right],\tag{5.23}$$

given in terms of (5.18)-(5.20).

In order to compute order parameters we need a canonical homology basis. For the order parameters a_k we have the basis A_k, as shown in Fig. 8.
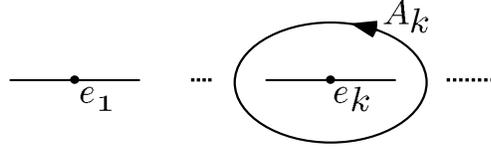

Figure 8

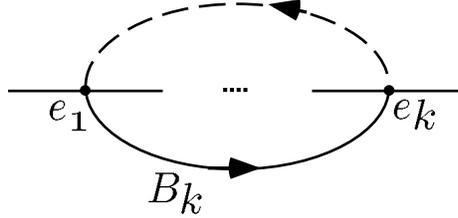We need the cycles B_k for the dual order parameters $a_{D,k}$ as shown in Fig. 9.

Figure 9

The cycle B_k connects sheets y_1 and y_2, with the solid line on sheet y_1 and the dashed line on y_2, with B_k passing through the branch-cut as shown. To compute (3.9), one only needs $(\lambda_1 - \lambda_2)$, so that we only need keep terms odd under $r \to -r$, with r as in (5.6).

The order parameter is

$$\begin{aligned}2\pi i\, a_k &= \oint_{A_k} \lambda \\ &\simeq \oint_{A_k} (\lambda_I + \lambda_{II} + \cdots)\end{aligned}$$

19

$$= \oint_{A_k} dx \left[\frac{x\left(\frac{A'}{A} - \frac{B'}{2B}\right)}{\sqrt{1 - \frac{B}{A^2}}} - L^6 \frac{\left(A - \frac{B}{2A}\right)}{B^2 \sqrt{1 - \frac{B}{A^2}}} \right]. \tag{5.24}$$

The second term does not contribute to $\mathcal{O}(L^2)$, as there are no poles at $x = e_k$. Thus to our order

$$a_k = e_k + L^2 \left[\frac{1}{4} \frac{\partial S_k}{\partial x}(e_k) - R_k(e_k) \right] + \cdots \tag{5.25}$$

The computation of the dual order parameter is considerably more complicated, with

$$2\pi i \, a_{D,k} = \oint_{B_k} (\lambda_I + \lambda_{II} + \cdots). \tag{5.26}$$

The hyperelliptic approximation to (5.26) gives

$$2\pi i \, (a_{D,k})_I = 2 \int_{x_1^-}^{x_k^-} dx \left[\frac{x\left(\frac{A'}{A} - \frac{B'}{2B}\right)}{\sqrt{1 - \frac{B}{A^2}}} \right] = \sum_{m=0}^{\infty} I_m, \tag{5.27}$$

where the series is obtained by expanding the square-root in the denominator of (5.27) with

$$I_m = \frac{2\Gamma\left(m + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(m+1)} \int_{x_1^-}^{x_k^-} dx \; x \left(\frac{A'}{A} - \frac{B'}{2B}\right) \left(\frac{B}{A^2}\right)^m. \tag{5.28}$$

Since we are only interested in computing $a_{D,k}$ accurate to $\mathcal{O}(L^2)$ [1-instanton order], and since $B = \mathcal{O}(L^2)$, one might naively expect that one need keep only $m = 0$ and $m = 1$ terms in (5.27)-(5.28). However, this is wrong, as the integrations produce terms with $(1/L)^p$, for $p = 1$ to ∞. Therefore we need consider all m in (5.27), and then sum the series. The result of computing (5.27)-(5.28) is

$$\begin{aligned}
2\pi i \, (a_{D,k})_I &= \oint_{B_k} \lambda_I = [N + 2 + (N+2)\log(-1) + 2\log L]\, a_k \\
&\quad -2\sum_{i \neq k}^{N}(a_k - a_i)\log(a_k - a_i) + \sum_{i=1}^{N}(a_k + a_i)\log(a_k + a_i) - 2a_k \log a_k \\
&\quad +L^2 \left[-\frac{1}{4}\frac{\partial S_0}{\partial x}(0)\log a_k - \frac{1}{4}\sum_{j=1}^{N}\frac{\partial S_j}{\partial x}(a_j)\log(a_k + a_j) \right. \\
&\quad \left. + \frac{1}{4}\frac{\partial S_k}{\partial x}(a_k) + \frac{1}{4}\frac{S_0(0)}{a_k} - \frac{1}{2}\sum_{i \neq k}^{N}\frac{S_i(a_i)}{a_k - a_i} \right] + \mathcal{O}(L^4).
\end{aligned} \tag{5.29}$$

20

The terms of $\mathcal{O}(L^2)$ are 1-instanton contributions to the dual order parameter. The presence of unallowed $L^2 \log a_k$ type terms indicates that (5.29) cannot be the complete contribution to $(a_{D,k})$ at 1-instanton order. Indeed, the second term in (5.26) is crucial for obtaining the correct result. This correction to the hyperelliptic approximation gives

$$
\begin{aligned}
(2\pi i a_{D,k})_{II} &= \oint_{B_k} \lambda_{II} \\
&= -2L^2 \int_{x_1^-}^{x_k^-} dx\, \frac{C(x)}{D^2(x)} \\
&= -L^2 \int_{x_1^-}^{x_k^-} dx \frac{\prod_i (x - e_i)}{x^2 \prod_i (x + e_i)^2} \\
&= \frac{1}{4} L^2 \left[\frac{\partial S_0}{\partial x}(0) \log a_k + \sum_{j=1}^{N} \frac{\partial S_j}{\partial x}(a_j) \log(a_k + a_j) + \frac{S_0(0)}{a_k} + \sum_{j=1}^{N} \frac{S_j(a_j)}{a_k + a_j} \right] .
\end{aligned}
$$

(5.30)

Notice that the $L^2 \log a_k$ type terms in (5.30) cancel those in (5.29). The final result for $a_{D,k}$ to our order is given by (5.29) and (5.30), with

$$
\begin{aligned}
2\pi i a_{D,k} &= [N + 2 + (N + 2) \log(-1) + 2 \log L]\, a_k \\
\\
&\quad -2 \sum_{i \neq k} (a_k - a_i) \log(a_k - a_i) + \sum_i (a_k + a_i) \log(a_k + a_i) - 2a_k \log a_k \\
\\
&\quad + L^2 \left[\frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) + \frac{1}{2} \frac{S_0(0)}{a_k} - \frac{1}{2} \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} + \frac{1}{4} \sum_{j=1}^{N} \frac{S_j(a_j)}{a_k + a_j} \right] \\
\\
&\quad -(k \to 1),
\end{aligned}
$$

(5.31)

where the first and second rows of (5.31) give the classical and 1-loop result respectively. It is possible to write (5.31) as:

$$
2\pi i a_{D,k} = 2\pi i \frac{\partial}{\partial a_k}[\mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}}] + L^2 \frac{\partial}{\partial a_k}[-\frac{1}{2} S_0(0) + \frac{1}{4} \sum_{k=1}^{N} S_k(a_k)],
$$

(5.32)

21

in accord with the integrability required of the dual order parameter [cf (3.6)]. Thus we finally find for the prepotential for $SU(N)$ gauge theory with one massless antisymmetric hypermultiplet [6]

$$\mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}} = \frac{1}{4\pi i}\left[\frac{3}{2}(N+2) + (N+2)\log(-1) + 2\log 2\right]\sum_j a_j^2 \tag{5.33}$$

$$+\frac{i}{8\pi}\left[\sum_{i,j=1}^{N}(a_i - a_j)^2 \log\frac{(a_i - a_j)^2}{\Lambda^2} - \sum_{i<j}(a_i + a_j)^2 \log\frac{(a_i + a_j)^2}{\Lambda^2}\right],$$

and

$$\mathcal{F}_{1-\text{inst}} = \frac{1}{2\pi i}\left[-\frac{1}{2}S_0(0) + \frac{1}{4}\sum_k S_k(a_k)\right]. \tag{5.34}$$

Eq. (5.34) is a prediction of M-theory which may be tested against microscopic calculations. This is presently possible for $SU(N)$ with $N \leq 4$, since

$SU(2)$ + antisymmetric = $SU(2)$ (pure gauge theory);

$SU(3)$ + antisymmetric = $SU(3)$ + 1 fundamental;

$SU(4)$ + antisymmetric = $SO(6)$ + 1 fundamental.

In each of these three cases, (5.34) agrees with 1-instanton calculations from $\mathcal{L}_{\text{micro}}$ [26]. For $N \geq 5$, (5.34) should be regarded as predictions of M-theory, awaiting testing. The fact that (5.34) agrees with microscopic calculations, when available, after a long derivation, with distinct methods, is already impressive.

There are further applications of hyperelliptic perturbation theory, where the analysis is very similar to that sketched above. For $SU(N)$ gauge theory with a hypermultiplet in the symmetric representation, which is also described by a cubic SW curve [4], one obtains [7]

$$\mathcal{F}_{1-\text{inst}} = \frac{1}{8\pi i}\sum_{k=1}^{N} S_k(a_k), \tag{5.35}$$

where

$$S_k(a_k) = \frac{4(-1)^N a_k^2 \prod_{i=1}^{N}(a_k + a_i)}{\prod_{i \neq k}^{N}(a_k - a_i)^2}. \tag{5.36}$$

One may also add hypermultiplets in the fundamental representation. Moreover, one may consider hypermultiplets with non-zero masses. For $SU(N)$ gauge theory with an antisymmetric representation and $N_f < N + 2$, which is described by a cubic SW curve [4], one finds [8]:

$$2\pi i \mathcal{F}_{1-\text{inst}} = \frac{1}{4} \sum_{k=1}^{N} S_k(a_k) - \frac{1}{2} S_0(0), \tag{5.37}$$

where

$$S_k(a_k) = \frac{4(-1)^N \prod_{j=1}^{N_f}(a_k + M_j) \prod_{i=1}^{N}(a_k + a_i + m)}{(a_k + \frac{1}{2}m)^2 \prod_{i \neq k}^{N}(a_k - a_i)^2}, \tag{5.38}$$

$$S_0(0) = \frac{4(-1)^N \prod_{j=1}^{N_f}(M_j - \frac{1}{2}m)}{\prod_{k=1}^{N}(a_k + \frac{1}{2}m)}, \tag{5.39}$$

where M_j (m) is the mass of the hypermultiplet in the fundamental (resp. antisymmetric) representation. Eqs. (5.38) and (5.39) agree with scaling limits taking M_j and/or $m \to \infty$. Eqs. (5.37)–(5.39) provide additional tests of M-theory, since

$SU(2)$ + antisymmetric + N_f fundamentals = $SU(2)$ + N_f fundamentals,

$SU(3)$ + antisymmetric + N_f fundamentals = $SU(3)$ + $(N_f + 1)$ fundamentals,

both of which agree with microscopic instanton calculations [26]. However, for $SU(N)$ + antisymmetric + N_f fundamentals with $N \geq 4$, (5.37)–(5.39) are predictions of M-theory which are as yet untested.

The last example of an $SU(N)$ theory with a cubic SW curve is $SU(N)$ + symmetric + N_f fundamentals, with $N_f < N + 2$. Here the result is [8]:

$$2\pi i \mathcal{F}_{1-\text{inst}} = \frac{1}{4} \sum_{k=1}^{N} S_k(a_k), \tag{5.40}$$

where

$$S_k(a_k) = \frac{4(-1)^N (a_k + \frac{1}{2}m)^2 \prod_{j=1}^{N_f}(a_k + M_j) \prod_{i=1}^{N}(a_k + a_i + m)}{\prod_{i \neq k}^{N}(a_k - a_i)^2}. \tag{5.41}$$

Although (5.40)–(5.41) have the correct scaling limits as m or $M_j \to \infty$, these remain predictions of M-theory which have not been tested as yet.

We have seen that whenever the predictions of the cubic SW curves obtained from M-theory have been tested, agreement has been found with those of microscopic calculations. However, there remain numerous further opportunities to subject M-theory predictions to testing.

In the next section we discuss aspects of the "universality" of the results of this section, and use those observations to construct a curve which is not yet obtainable from M-theory.

6. Universality

In addition to the results for cubic SW curves discussed in the previous section, eqs. (5.34)-(5.41), we also wish to recall the results for a hyperelliptic curve, $\mathrm{SU}(N) + N_f$ fundamentals, with $N_f < 2N$. That is [20],

$$8\pi i \mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k), \tag{6.1}$$

where

$$S_k(a_k) = \frac{\prod_{j=1}^{N_f}(a_k + M_j)}{\prod_{i \neq k}^{N}(a_k - a_i)^2}. \tag{6.2}$$

If now one examines the cases treated in the previous section, together with (6.1)–(6.2), one observes certain universal features:

(i) The natural variables for this class of problems are the order parameters $\{a_k\}$ and not the gauge invariant moduli.

(ii) The 1-instanton contribution to the prepotential can be written as [7, 8, 20]

$$8\pi i \mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k), \tag{6.3}$$

for $\mathrm{SU}(N) + N_f$ fundamentals or $\mathrm{SU}(N) +$ symmetric $+ N_f$ fundamentals, and

$$8\pi i \mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_m(-m), \tag{6.4}$$

24

for $SU(N)$ + antisymmetric + N_f fundamental hypermultiplets [6, 8], where the second term in (6.4) removes a spurious singularity in $\mathcal{F}_{1-\text{inst}}$ as $a_k \to -m$. The generalization of (6.4) will be important when we construct $\mathcal{F}_{1-\text{inst}}$ for $SU(N)$ gauge theory with two antisymmetric hypermultiplets.

We define (a posteriori) $S(x)$ which generalizes (5.19) as

$$S(x) = \frac{S_k(x)}{(x - a_k)^2} = \frac{S_m(x)}{(x + m)^2}. \tag{6.5}$$

Using the results of Sec. 5, and of (6.2), we tabulate the resulting $S(x)$ in the first three entries of Table 1. (We will discuss the 4th row of Table 1 shortly). It should be noted that Table 1 includes all generic cases of asymptotically free $\mathcal{N} = 2$ $SU(N)$ gauge theories.

A careful examination of the first three rows of Table 1 leads to the following empirical rules for $S(x)$. $S(x)$ is given as the product of the following factors, each corresponding to a different $\mathcal{N} = 2$ multiplet in a given representation of $SU(N)$:

(1) Pure gauge multiplet factor

$$\frac{1}{\prod_{i=1}^{N}(x - a_i)^2}. \tag{6.6}$$

(2) Fundamental representation. A factor

$$(x + M_j) \tag{6.7}$$

for each hypermultiplet of mass M_j in the fundamental representation.

(3) Symmetric representation. A factor

$$(-1)^N (x + m)^2 \prod_{i=1}^{N}(x + a_i + 2m) \tag{6.8}$$

for each hypermultiplet of mass $2m$ in the symmetric representation.

(4) Antisymmetric representation. A factor

$$\frac{(-1)^N}{(x + m)^2} \prod_{i=1}^{N}(x + a_i + 2m) \tag{6.9}$$

for each hypermultiplet of mass $2m$ in the antisymmetric representation.

From these empirical rules, we predict $S(x)$ for $\mathrm{SU}(N) + 2$ antisymmetric $+ N_f$ fundamentals, with $N_f \leq 4$, as shown in the last entry of Table 1.

Hypermultiplet Representations	$S(x)$
N_f fundamentals (ref. [20])	$\dfrac{4\prod_{j=1}^{N_f}(x+M_j)}{\prod_{i=1}^{N}(x-a_i)^2}$
1 symmetric $+N_f$ fundamentals (ref. [7, 8])	$\dfrac{4(-1)^N(x+m)^2\prod_{i=1}^{N}(x+a_i+2m)\prod_{j=1}^{N_f}(x+M_j)}{\prod_{i=1}^{N}(x-a_i)^2}$
1 antisymmetric $+N_f$ fundamentals (ref. [6, 8])	$\dfrac{4(-1)^N\prod_{i=1}^{N}(x+a_i+2m)\prod_{j=1}^{N_f}(x+M_j)}{(x+m)^2\prod_{i=1}^{N}(x-a_i)^2}$
2 antisymmetric $+N_f$ fundamentals (ref. [10])	$\dfrac{4\prod_{i=1}^{N}(x+a_i+2m_1)\prod_{i=1}^{N}(x+a_i+2m_2)\prod_{j=1}^{N_f}(x+M_j)}{(x+m_1)^2(x+m_2)^2\prod_{i=1}^{N}(x-a_i)^2}$

Table 1: The function $S(x)$ for $\mathrm{SU}(N)$ gauge theory, with different matter content. The hypermultiplets in the fundamental representation have masses M_j. The symmetric or antisymmetric representation has mass $2m$. If there are two antisymmetric representations, their masses are $2m_1$ and $2m_2$.

Given this $S(x)$, we then predict [10]

$$8\pi i \mathcal{F}_{1-\mathrm{inst}} = \sum_{k=1}^{N} S_k(a_k) - 2S_{m_1}(-m_1) - 2S_{m_2}(-m_2), \qquad (6.10)$$

where $S_k(a_k)$ and S_m are constructed from (6.5) and the 4th entry of Table 1. The last two terms of (6.10) remove the spurious singularities as $a_k \to -m_1$ and $a_k \to -m_2$, which generalizes (6.4). Thus, from the observed regularities, we are able to predict $\mathcal{F}_{1-\text{inst}}$ for $SU(N) + 2$ antisymmetric $+ N_f$ fundamentals, with $N_f \leq 3$, even though no SW curve is available from M-theory!

The predictions of Table 1 and (6.10) can be tested as follows:

1) $SU(2) + 2$ antisymmetric $+ (N_f \leq 3) = SU(2) + (N_f \leq 3)$.

2) $SU(3) + 2$ antisymmetric $+ (N_f \leq 3) = SU(3) + (N_f \leq 5)$.

3) Limit m_1 or $m_2 \to \infty$ reduces to $SU(N) + $ antisymmetric $+ (N_f \leq 3)$.

In each of these 3 cases, our predicted $\mathcal{F}_{1-\text{inst}}$ finds agreement.

The program we describe in the next section involves:

1) Assume that $\mathcal{F}_{1-\text{inst}}$ from Table 1 and equation (6.10) are correct.

2) Find a SW curve which gives

$$\mathcal{F} = \mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}} + \mathcal{F}_{1-\text{inst}}.$$

3) Impose consistency with M-theory.

It should be emphasized that there is no known derivation of the empirical rules of (6.6)-(6.9). This is a problem that deserves consideration from first principles.

7. Reverse Engineering a Curve

Although there is no known SW curve for $SU(N)$ gauge theory with two antisymmetric representations and $N_f \leq 3$ hypermultiplets, one can attempt to reverse engineer a curve from the information in Table 1 and (6.10). The strategy for the construction is

1) $\mathcal{F}_{\text{classical}} + \mathcal{F}_{1-\text{loop}}$ from perturbation theory.

2) $\mathcal{F}_{1-\text{inst}}$ as predicted in Table 1 and (6.10).

3) These two steps imply that $a_{D,k} = \frac{\partial \mathcal{F}}{\partial a_k}$ is known to 1-instanton accuracy.

4) Reproduce this expression from period integrals of a Riemann surface, to be constructed from the above data.

5) Ensure that the proposed Riemann surface is consistent with M-theory.

To begin with we postulate a quartic curve

$$y^4 + \cdots = 0$$

Why? Since a cubic curve is needed for $SU(N)$ gauge theory with a hypermultiplet in the symmetric or antisymmetric representations [4], at least a cubic curve is required. Further, Witten has shown [3] that for $SU(N) \times SU(N) \times \overset{m \text{ factors}}{\cdots} \times SU(N)$ the corresponding curve is

$$y^{m+1} + \cdots = 0,$$

which results from $m+1$ parallel NS 5-branes, and N D4-branes suspended between neighboring pairs of NS 5-branes. However, for $(SU(N))^m$, with $m \geq 3$, we have shown [9] that to attain 1-instanton accuracy, one *only* needs a quartic approximation

$$y^4 + \cdots = 0,$$

to the full y^{m+1} curve. Therefore, we only need a quartic curve if we are trying to reproduce the prepotential to 1-instanton accuracy. One can also show that the most general quartic curve consistent with M-theory is of the form [3, 10]

$$
L^4 j_1(x) P_2(x) t^2 + L P_1(x) t + P_0(x) + L j_0(x) P_{-1}(x) \frac{1}{t}
$$
$$
+ L^4 j_0{}^2(x) j_{-1}(x) P_{-2}(x) \frac{1}{t^2} = 0, \tag{7.1}
$$

where $j_n(x)$ are associated to the N_f flavors in the fundamental representation, and $P_n(x)$ are associated to the positions of D4-branes. For our problem,

$$L^2 = \Lambda^{4-N_f}. \tag{7.2}$$

Let us regard (7.1) as the quartic approximation to a curve of infinite order. (We will justify this shortly when we demand that the brane picture for (7.1) be consistent with M-theory, and an infinite number of reflections). Thus, (7.1) is a truncation of

$$\sum_{n=1}^{\infty} L^{n^2} \prod_{s=1}^{n-1} j_s{}^{n-s}(x)\, P_n(x)\, t^n \;+\; P_0(x) \;+\; \sum_{n=1}^{\infty} L^{n^2} j_0{}^n(x) \prod_{s=1}^{n-1} j_{-s}{}^{n-s}(x)\, P_{-n}(x)\, t^{-n} = 0.$$

$$(7.3)$$

There should be a symmetry between the two antisymmetric representations, which must appear in the curve. This is manifest by the symmetries

$$
\begin{aligned}
P_n(x; m_1, m_2) &= P_{-n}(x; m_2, m_1), \\
j_n(x; m_1, m_2) &= j_{-n}(x; m_2, m_1).
\end{aligned}
$$

$$(7.4)$$

The curve is then invariant under the involution

$$t \longrightarrow \frac{j_0(x)}{t}; \qquad m_2 \longleftrightarrow m_1. \tag{7.5}$$

We wish to begin with the hyperelliptic perturbation expansion solutions to (7.1), which is facilitated by the change of variables

$$t = \frac{1}{L P_1(x)}. \tag{7.6}$$

The curve becomes

$$\frac{L^2\, j_1\, P_2}{P_1{}^2}\, y^4 + y^3 + P_0\, y^2 + L^2 j_0 P_1 P_{-1} y + L^6 j_0^2 j_{-1} P_1^2 P_{-2} \;=\; 0, \tag{7.7}$$

which is of the form

$$\epsilon_1(x) y^4 + y^3 + 2A(x) y^2 + B(x) y + \epsilon_2(x) = 0. \tag{7.8}$$

29

To first order in ϵ_1 and ϵ_2, the solution for y_1 (sheet 1) is

$$
\begin{aligned}
y_1 &= -(A+r) - \frac{(A+r)^3}{2r}\epsilon_1 - \frac{1}{2r(A+r)}\epsilon_2 + \cdots \\
&= (y_1)_I + (y_1)_{II} + \cdots,
\end{aligned}
\tag{7.9}
$$

where

$$
y_I = -(A+r) , \qquad r = \sqrt{A^2 - B}.
\tag{7.10}
$$

Similarly, the SW differential is

$$
\lambda = \frac{x\,dy}{y} = \lambda_I + \lambda_{II} + \cdots
\tag{7.11}
$$

The problem is to find $P_n(x)$ and $j_n(x)$.

From the period integral

$$
\oint_{A_k} \lambda_I = \oint_{A_k} dx \frac{\left(\frac{A'}{A} - \frac{B'}{2B}\right)}{\sqrt{1 - \frac{B}{A^2}}},
\tag{7.12}
$$

one has

$$
a_k = e_k + \mathcal{O}(L^2),
\tag{7.13}
$$

for the order parameters. We *know* the dual order parameters, to 1-instanton accuracy, from Table 1 and (6.10). They are related to the unknown coefficients as follows from the hyperelliptic approximation:

$$
\begin{aligned}
2\pi i (a_{D,k})_I &= \oint_{B_k} \lambda_I
&& (7.14) \\
&= 2L^2 \int_{x_1^-}^{x_k^-} dx \left[\frac{j_0(x) P_{-1}(x) P_1(x)}{P_0^2(x)} + \cdots\right]
&& (7.15) \\
&= -\frac{1}{2} L^2 \sum_{i \neq k}^{N} \frac{S_i(a_i)}{(a_k - a_i)} + \cdots
&& (7.16) \\
&= \frac{L^2}{2} \int_{x_1^-}^{x_k^-} dx \left[\sum_{i=1}^{N} \frac{S_i(x)}{(x - e_i)^2} + \cdots\right].
&& (7.17)
\end{aligned}
$$

30

Eq. (7.15) comes from the postulated form of the curve (7.7) and (7.8), while (7.16) is obtained from Table 1 and (6.10), and (7.17) expresses (7.16) as a period integral. There are other pieces of (7.14)-(7.17) not shown. We emphasize those features which lead to a solution for P_n and j_n. Comparing (7.15) with (7.17) leads to the postulate

$$\frac{j_0(x)P_{-1}(x)P_1(x)}{P_0^2(x)} = \frac{1}{4}S(x) + \mathcal{O}(L^2), \tag{7.18}$$

where the $\mathcal{O}(L^2)$ is what we call *subleading* terms.

The correction to the hyperelliptic approximation gives

$$\begin{aligned}
2\pi i(a_{D,k})_{II} &= 2\int_{x_1^-}^{x_k^-} \lambda_{II} \\
&= -L^2 \int_{x_1^-}^{x_k^-} dx \left[\frac{j_1 P_0 P_2}{P_1^2} + \frac{j_{-1} P_0 P_{-2}}{P_{-1}^2}\right]. \tag{7.19}
\end{aligned}$$

Note that the same combination of coefficients appears in (7.18) and (7.19), but with the subscripts shifted as $n \to n+1$ or $n-1$. The choice

$$\begin{aligned}
\frac{j_1 P_0 P_2}{P_1^2} &= \frac{1}{4}S(-x-2m_2) + \mathcal{O}(L^2), \\
\frac{j_{-1} P_0 P_{-2}}{P_{-1}^2} &= \frac{1}{4}S(-x-2m_1) + \mathcal{O}(L^2), \tag{7.20}
\end{aligned}$$

reproduces a number of the terms of $(a_{D,k})$, with those remaining terms not obtained from (7.14)-(7.17) or (7.19)-(7.20) assumed to come from subleading terms. Comparing (7.20) with (7.18) we argue that

$$\frac{j_n P_{n-1} P_{n+1}}{P_n^2} = \frac{1}{4}S(\text{known reflection and shift in } x) + \mathcal{O}(L^2). \tag{7.21}$$

This will have important geometrical consequences for the M-theory picture!

Remarkably, (7.18), (7.20) and (7.21) can be uniquely solved for the *leading* coefficient functions, with the first few

$$P_0(x) = \prod_{i=1}^{N}(x-a_i) + \mathcal{O}(L^2),$$

$$P_1(x) = (-1)^N (x + m_2)^{-2} \prod_{i=1}^{N} (x + a_i + 2m_2) + \mathcal{O}(L^2),$$

$$P_2(x) = (x + m_2)^{-6}(x + 2m_2 - m_1)^{-2} \prod_{i=1}^{N} (x - a_i + 2m_2 - 2m_1) + \mathcal{O}(L^2), \quad (7.22)$$

and

$$j_0(x) = \prod_{j=1}^{N_f} (x + M_j),$$

$$j_1(x) = (-1)^{N_f} \prod_{j=1}^{N_f} (x + 2m_2 - M_j), \quad (7.23)$$

together with (7.4). We shall associate the numerators of the $P_n(x)$ with the positions of the D4-branes, the denominators to that of $O6^-$ orientifold planes, and $j_n(x)$ to the positions of the D6-branes.

We now argue that (7.1) is incomplete if consistency with M-theory is demanded, with the result of a curve of infinite order, and therefore an infinite chain of NS 5-branes and orientifolds. To see the origin of this assertion, recall that the brane picture for SU(N) gauge theory with an antisymmetric representation of mass m is shown in Fig. 10, showing only the NS 5-branes and the $O6^-$ plane for clarity.
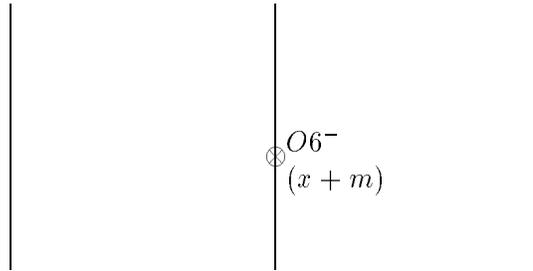
Figure 10

Therefore, to begin with, for SU(N) gauge theory with two antisymmetric representations of masses m_1 and m_2, we expect *at least* the brane structure in Fig. 11.
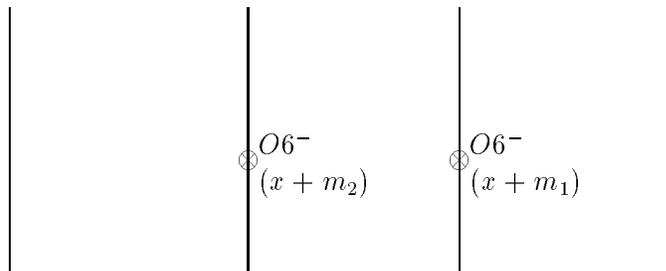
Figure 11

Again in Fig. 11, only the NS 5-branes and $O6^-$ are shown, while the N D4-branes connecting the NS 5-branes and the flavor D6-branes are not shown for clarity. The first observation is that to satisfy all possible mirrors, one must have an infinite chain of NS 5-branes and $O6^-$ orientifolds, since one must satisfy the reflections in *each* of the $O6^-$ orientifold planes separately. A portion of this chain is shown in Fig. 12, which differs from Fig. 11 in that the positions of D4-branes and D6-(flavor) branes are shown. Fig. 13 is an expanded version of the chain showing six NS 5-branes. One can check that all the necessary mirrors about any given $O6^-$ orientifold plane are satisfied.

There are two important observations:

1) If $m_2 \to \infty$, most of the D4-branes, D6-branes and $O6^-$ planes slide off to infinity, leaving us with the configuration of Fig. 14, which coincides with Fig. 10 for $SU(N)$ and an antisymmetric representation of mass m_1. This is the geometric analogue of the double scaling limit by which we checked $\mathcal{F}_{1-\text{inst}}$ for $S(x)$ and (6.10). (Of course, we equally could have taken $m_1 \to \infty$.)

2) The ratio (7.21) is characteristic of a single cell of two adjacent NS 5-branes, and N linking D4-branes. The statement that the ratio (7.21) is $\frac{1}{4}S(x)$ up to a known reflection and shift in x is equivalent to saying that one can begin the hyperelliptic approximation with *any* cell in the infinite chain of NS 5-branes.

Thus the infinite chain of NS 5-branes and $O6^-$ orientifolds, a portion of which is shown in

Fig. 13, is equivalent to the curve (7.3), with solution (7.21)-(7.23), etc. The question arises whether the infinite series can be summed. For simplicity we consider $\mathrm{SU}(N)$ with two anti-symmetric representations and $N_f = 0$, with $m_1 = m_2 = m$. The series (7.3), with coefficients $P_n(x)$ known to *leading* order in L, and $j_n(x) = 0$, can be summed. These leading order terms sum to

$$H_0(x) \sum_{n=-\infty}^{\infty} e^{2\pi i \tau(x) n^2} t^{2n} + H_1(x) \sum_{n=-\infty}^{\infty} e^{2\pi i \tau(x)(n+1/2)^2} t^{2n+1} = 0, \tag{7.24}$$

where

$$
\begin{aligned}
H_0(x) &= \prod_{i=1}^{N}(x - a_i - m), \\
H_1(x) &= H_0(-x), \\
e^{2\pi i \tau(x)} &= \frac{L^4}{x^8}.
\end{aligned}
\tag{7.25}
$$

where the definition of x has been shifted by m relative to eq. (7.3). Eq. (7.24) can be reexpressed in terms of Jacobi theta functions as

$$H_0(x)\,\theta_3(2\nu|2\tau(x)) + H_1(x)\,\theta_2(2\nu|2\tau(x)) = 0, \tag{7.26}$$

where $t = e^{2\pi i \nu}$, and [27]

$$
\begin{aligned}
\theta_1(\nu|\tau) &= i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau(n-\frac{1}{2})^2} e^{2\pi i \nu(n-\frac{1}{2})}, \\
\theta_2(\nu|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n-\frac{1}{2})^2} e^{2\pi i \nu(n-\frac{1}{2})}, \\
\theta_3(\nu|\tau) &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{2\pi i \nu n}, \\
\theta_4(\nu|\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau n^2} e^{2\pi i \nu n}.
\end{aligned}
\tag{7.27}
$$

Note that (7.26) only includes the leading terms of the curve. Since $\mathrm{Im}\,\tau(x) > 0$ is required for the theta functions, the series is not well defined for L large or $x \to 0$, the latter describing

34

the approach to an orientifold. This clearly indicates the need for subleading terms (*i.e.* terms of higher order in $e^{2\pi i \tau(x)}$ for a given power of t) in order to "resolve" the singularities at the orientifold. These are non-perturbative issues. Also note that (7.26) is reminiscent of, but not identical to, the spectral curve of the Calogero-Moser model, which describes $SU(N)$ with an adjoint matter hypermultiplet [18, 22].

Another case for which the infinite series (7.3) can be summed is that for which we extrapolate to $N_f = 4$, and take $m_1 = m_2 = M_j$, $(j = 1, ..., 4)$. This is an *elliptic* model, of which more will be discussed in the next section. Using our solution for the coefficients $P_n(x)$ and $j_n(x)$ we obtain

$$H_0(x)\,\theta_3(2\nu|2\tau) \;+\; H_1(x)\,\theta_2(2\nu|2\tau) \;=\; 0, \qquad\qquad (7.28)$$

where now τ is independent of x, and is the modular parameter of the elliptic model. Thus, we are dealing with a mass-deformed scale-invariant case, *i.e.* a theory with zero beta function. We return to this case in the next section, where we indicate in (8.13)–(8.17) that there are no subleading terms in (7.28).
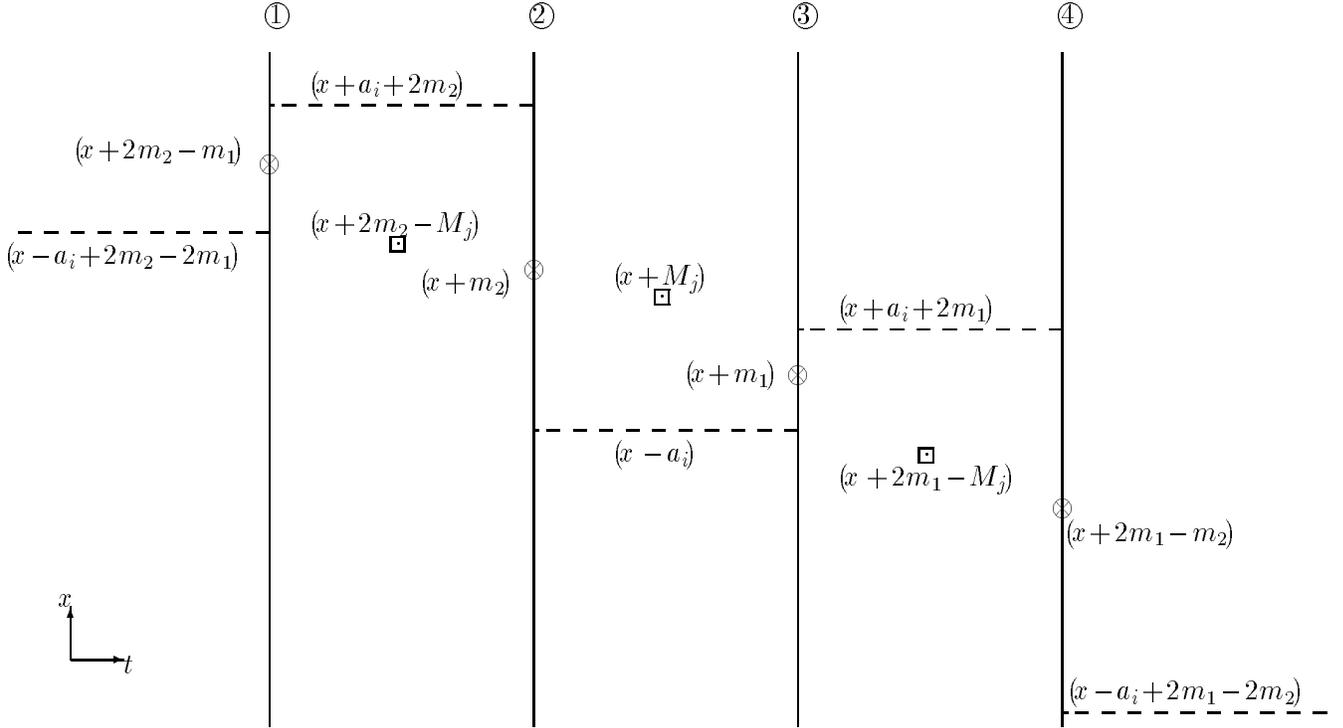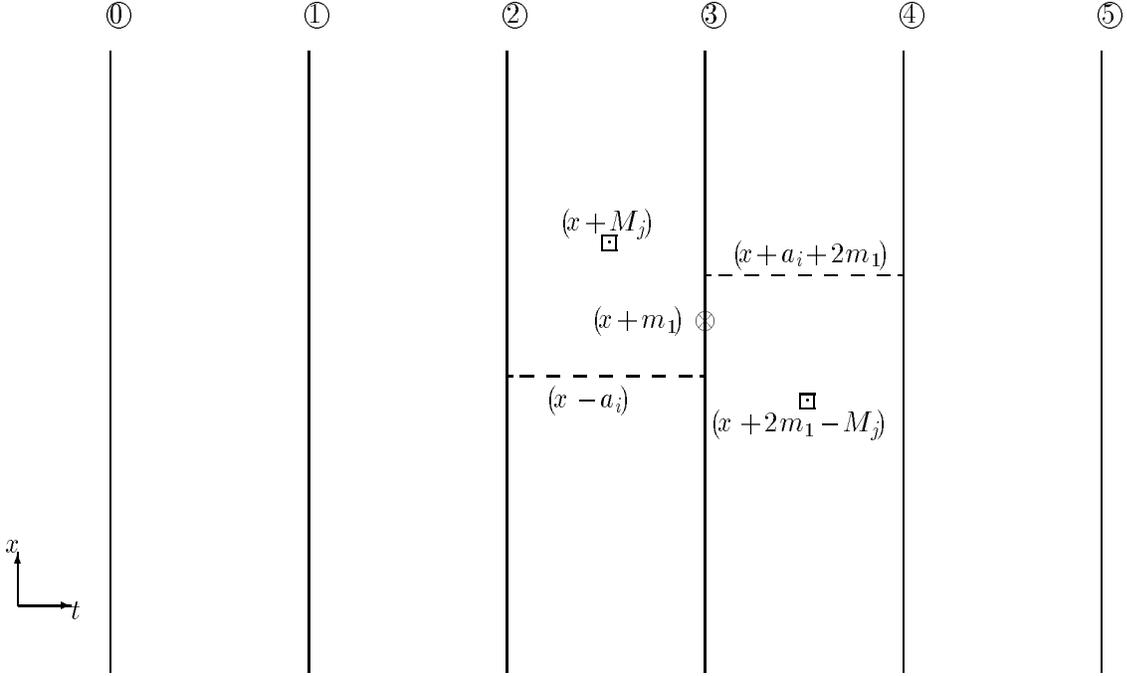
Figure 12:

1) *vertical lines*: parallel, equally spaced NS 5-branes.

2) *dashed lines*: N parallel D4-branes connect pairs of adjacent NS 5-branes.

3) $\otimes$: $O6^-$ orientifold planes.

4) $\boxdot$: D6-(flavor) branes.

Due to mirrors, the picture must extend infinitely to right and left.

36
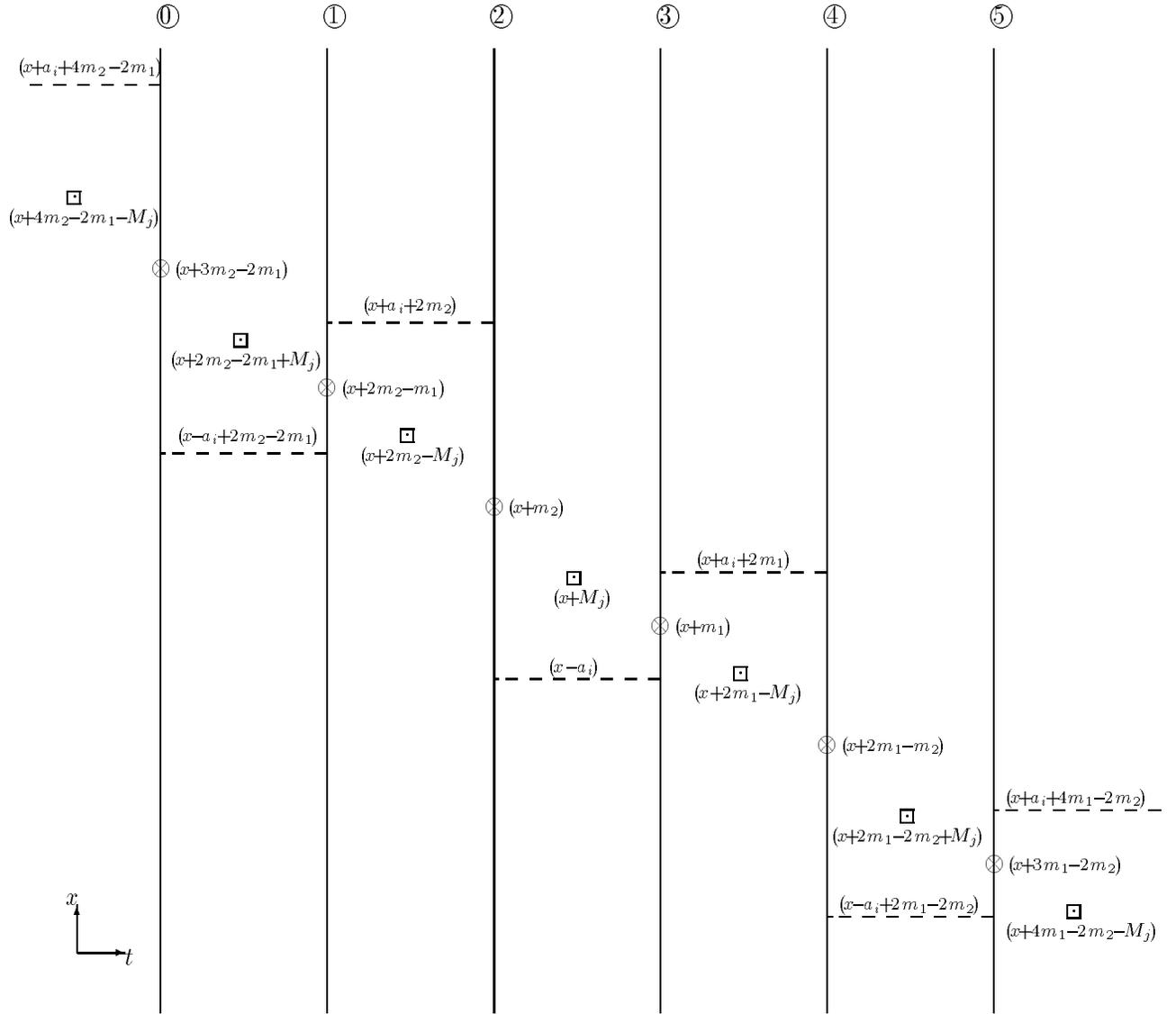
Figure 13: An expanded version of Figure 12 with six NS 5-branes. One can do hyperelliptic approximation about *any* pair of adjacent NS 5-branes.

Figure 14: The $m_2 \to \infty$ limit of Figure 13. In this limit, only the NS 5-branes ②, ③, and ④ remain connected by D4-branes. The other D4-branes and $O6^-$ planes have "slid off" to $x \sim \infty$. It agrees with the M-theory picture of 1 antisymmetric hypermultiplet with mass $2m_1$.

8. Elliptic Models

We remarked that (7.28) was reminiscent of the Calogero-Moser model. Let us consider this in more detail, by examining $\mathcal{N} = 2$ SU(N) gauge theory with an adjoint hypermultiplet of mass m. This scale invariant model can be expressed in terms of an M-theory picture [3]:
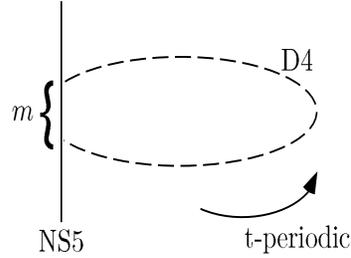

Figure 15

In Fig. 15, there are N D4-branes suspended between a single NS 5-brane, with a periodicity in t, but with a shift in v an ammount m for each circuit of t. Thus, there is a global mass m. The covering space of the S^1 (the t-variable) is shown in Fig. 16.
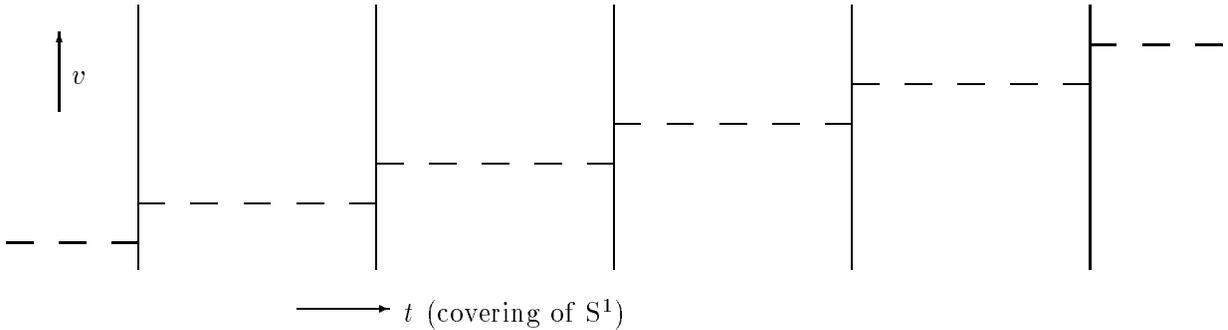

Figure 16

Generalizing the rules of Table 1, we expect from Fig. 16 the function

$$S(v) = \frac{\prod_{i=1}^{N}(v - a_i - m) \prod_{i=1}^{N}(v - a_i + m)}{\prod_{i=1}^{N}(v - a_i)^2}, \tag{8.1}$$

which has the property expressed in (7.21), but with no subleading terms. The instanton

expansion of $SU(N)$ + massive adjoint representation is given in [22, 28].

Witten [3] shows that the curve for this model is precisely that derived by Donagi and Witten [11] in the context of the integrable Hitchin system

$$F(v, x, y) = \sum_{j=0}^{N} A_j P_{N-j}(v), \tag{8.2}$$

where A_j are gauge invariant polynomials in the v (the vev of the scalar field), and where x and y are related by the elliptic curve

$$y^2 = (x - e_1)(x - e_2)(x - e_3). \tag{8.3}$$

They show that

$$P_n(v) = \sum_{i=0}^{n} \binom{n}{i} f_i \, v^{n-i}. \tag{8.4}$$

It can be shown that (8.2) can be reexpressed as

$$F(v, x, y) = \sum_{j=0}^{N} \frac{m^j}{j!} \, f_j \, H^{(j)}(t), \tag{8.5}$$

where

$$H(v) = \prod_{i=1}^{N} (v - a_i) \, , \qquad H^{(j)}(v) = \frac{d^j H(v)}{dv^j}. \tag{8.6}$$

Explicit calculation gives

$$
\begin{aligned}
f_0 &= 1 \, , \qquad f_1 = 0 \, , \qquad f_2 = -x \, , \\
f_3 &= 2y \, , \qquad f_4 = -3x^2 + 4x \sum_{i=1}^{3} e_i \, , \qquad \text{etc.}
\end{aligned}
\tag{8.7}
$$

It is claimed that this is equivalent to the Calogero-Moser model in ref. [22], for which the spectral curve is

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} e^{nz} H(k - nm) = 0, \tag{8.8}$$

40

where z parametrizes the torus with the identifications $z = z - 2\pi i = z - 2\pi i\tau$ and $q = e^{2\pi i\tau}$. Heretofore the connection between these curves has not been made explicit. We will sketch our results in this regard [29]. The curve (8.8) can be recast as

$$\sum_{j=0}^{N} \frac{(-m)^j}{j!} h_j(z) H^{(j)}(k - \tfrac{1}{2}m) = 0, \tag{8.9}$$

where

$$h_j(z) = \frac{1}{\theta_1(\frac{z}{-2\pi i}|\tau)} \frac{\partial^j}{\partial z^j} \theta_1\left(\frac{z}{-2\pi i}\bigg|\tau\right), \tag{8.10}$$

with θ_1 defined in eq. (7.27). Upon a change of variables, equation (8.9) becomes

$$\sum_{j=0}^{N} \frac{m^j}{j!} \tilde{f}_j(z) \, H^{(j)}(v) = 0. \tag{8.11}$$

The functions f_j and $\tilde{f}_j$ can be shown to be equivalent by using the connection

$$x = \wp(z), \qquad 2y = \wp'(z), \tag{8.12}$$

where $\wp(z)$ is the Weierstrass elliptic function.

Another important elliptic model is that of $SU(2N)$ with two antisymmetric representations and $N_f = 4$, with masses satisfying $m_1 = m_2 = M_j = m$, as in (7.28). This is an untwisted elliptic model with zero global mass. The brane picture has been discussed by Uranga [12], and
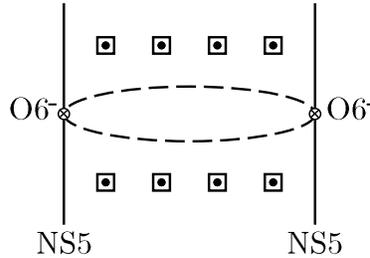
Figure 17

is shown in Fig. 17. The curve appropriate to Fig. 17 has been given by Gukov and Kapustin [13] as

$$v^{2N} + f_1(x, y)v^{2N-1} + \cdots + f_{2N}(x, y) = 0, \tag{8.13}$$

The coefficient functions are given by

$$
\begin{aligned}
f_{2j}(x, y) &= A_j, \\
f_{2j-1}(x, y) &= \frac{yB_j}{(x - e_3)} = \frac{(x - e_1)(x - e_2)}{y}B_j,
\end{aligned}
\tag{8.14}
$$

where x and y satisfy (8.3) and A_j and B_j are constants. We have shown [29] that (8.13) and (8.14) may be recast as

$$
\begin{aligned}
&H_+(v)\tan(\pi\nu) \prod_{n=1}^{\infty}(1 - 2e^{i\pi n\tau}\cos 2\pi\nu + e^{2\pi in\tau}) \\
&+ \quad H_-(v)[\theta_4(0|\tau)]^2 \prod_{n=1}^{\infty}(1 + 2e^{i\pi n\tau}\cos 2\pi\nu + e^{2\pi in\tau}) = 0,
\end{aligned}
\tag{8.15}
$$

with

$$
\begin{aligned}
H_\pm(v) &= \tfrac{1}{2}(H_0 \pm H_1), \\
H_0(v) &= \prod_{j=1}^{2N}(v - a_j - m), \\
H_1(v) &= H_0(-v).
\end{aligned}
\tag{8.16}
$$

We then found that (8.15) is equivalent to

$$H_0(v)\,\theta_3(2\nu|2\tau) + H_1(v)\,\theta_2(2\nu|2\tau) = 0, \tag{8.17}$$

after a suitable change of variables. This is *identical* to (7.28).

Thus, beginning with an elliptic model, and only analytic tools, we have a very dramatic confirmation of the reasoning of Sec. 7, which involved both analytic and geometric reasoning!

42

There are other interesting elliptic models, but we do not discuss these here.

On general principles, one expects there to be an integrable model associated to the solution of the $\mathcal{N} = 2$ SW problem (see Gorsky et. al. [14] for known examples). For example, SU(N) with adjoint matter was discussed in the first part of this section, and the explicit connection between the spectral curve of the Calogero-Moser model and the Donagi-Witten construction was sketched. However, there are a number of SW problems for which there are no known integrable models. A typical case in point is that of SU(N) with two antisymmetric representations and $N_f = 4$, with a single common mass m, discussed in the second part of this section. It would be very helpful to make these identifications, both on fundamental grounds, and also because the techniques of the integrable models may be able to provide new non-perturbative results, which go beyond hyperelliptic perturbation theory, carrying us into other regions of moduli space.

9. Concluding Remarks

There are a number of open problems which should be addressed. An incomplete list is:

1) Compute $\mathcal{F}_{1-\text{inst}}$ from $\mathcal{L}_{\text{micro}}$ for all the cases described in Table 1, so as to extend the test of M-theory. In every case where a test can be made, agreement has been found.

2) Explain *group-theoretically* the entries for $S(x)$ in Table 1, and the rules (6.6)-(6.9) abstracted from this table.

3) Study other elliptic models along the lines described in this review.

4) Find the associated integrable models for all the cases describable by a SW curve.

5) Extend the predictions of non-hyperelliptic curves to regions of moduli space for which the hyperelliptic perturbation theory is not valid.

As we have discussed, $\mathcal{N} = 2$ SW theory presents many varied opportunities for testing M-theory predictions for gauge theories. These deserve to be explored further to increase our confidence in M-theory.

10. Acknowledgement

References

[1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, erratum, *ibid* B430 (1994) 485, hep-th/9407087; Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[2] A. Hanany and E. Witten, Nucl. Phys. B492 (1997) 152, hep-th/9611230;

N. Evans, C. Johnson, and A. Shapere, Nucl. Phys. B505 (1997) 251, hep-th/9703210;

A. Brandhuber, J. Sonnenschein, S. Theisen, and S. Yankielowicz, Nucl. Phys. B502 (1997) 125, hep-th/9704044; Nucl. Phys. B504 (1997) 175, hep-th/9705232;

K. Landsteiner, E. Lopez, and D. Lowe, Nucl. Phys. B507 (1997) 197, hep-th/9705199;

S. Terashima and S.-K. Yang, Phys. Lett. B430 (1998) 102, hep-th/9803014.

[3] E. Witten, Nucl. Phys. B500 (1997) 3, hep-th/9703166.

[4] K. Landsteiner and E. Lopez, Nucl. Phys. B516 (1998) 273, hep-th/9708118;

K. Landsteiner, E. Lopez, and D. Lowe, J. High Energy Phys. 07 (1998) 011, hep-th/9805158.

[5] A. Klemm, W. Lerche, S. Theisen, and S. Yankielowicz, Phys. Lett. B344 (1995) 169, hep-th/9411048;

P. Argyres and A. Faraggi, Phys. Rev. Lett. 73 (1995) 3931, hep-th/9411057;

M. Douglas and S. Shenker, Nucl. Phys. B447 (1995) 271, hep-th/9503163;

U. Danielsson and B. Sundborg, Phys. Lett. B358 (1995) 273, hep-th/9504102; Phys. Lett. B370 (1996) 83, hep-th/9511180;

P. Argyres and M. Douglas, Nucl. Phys. B448 (1995) 93, hep-th/9505062;

A. Hanany and Y. Oz, Nucl. Phys. B452 (1995) 73, hep-th/9505075;

P. Argyres, M. Plesser, and A. Shapere, Phys. Rev. Lett. B75 (1995) 1699, hep-th/9505100;

A. Klemm, W. Lerche, and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929, hep-th/9505150;

I. Pesando, Mod. Phys. Lett. A10 (1995) 1871, hep-th/9506139;

A. Brandhuber and K. Landsteiner, Phys. Lett. B358 (1995) 73, hep-th/9507008;

J. Minahan and D. Nemeshansky, Nucl. Phys. B464 (1996) 3, hep-th/9507032;

P. Argyres and A. Shapere, Nucl. Phys. B461 (1996) 437, hep-th/9509175;

A. Hanany, Nucl. Phys. B466 (1996) 85, hep-th/9509176;

M. Alishahiha, F. Ardalan, and F. Mansouri, Phys. Lett. B381 (1996) 446, hep-th/9512005;

M. Abolhasani, M. Alishahiha, and A. Ghezelbash, Nucl. Phys. B480 (1996) 279, hep-th/9606043.

[6] S. Naculich, H. Rhedin, and H. Schnitzer, Nucl. Phys. B533 (1998) 275, hep-th/9804105.

[7] I. Ennes, S. Naculich, H. Rhedin, and H. Schnitzer, Int. J. Mod. Phys. A14 (1999) 301, hep-th/9804151.

[8] I. Ennes, S. Naculich, H. Rhedin, and H. Schnitzer, Nucl. Phys. B536 (1998) 245, hep-th/9806144.

[9] I. Ennes, S. Naculich, H. Rhedin, and H. Schnitzer, Phys. Lett. B452 (1999) 260, hep-th/9901124.

[10] I. Ennes, S. Naculich, H. Rhedin, and H. Schnitzer, Nucl. Phys. B558 (1999) 41, hep-th/9904078.

[11] R. Donagi and E. Witten, Nucl. Phys. B460 (1996) 299, hep-th/9510101;

R. Donagi, alg-geom/9705010.

[12] A. Uranga, Nucl. Phys. B526 (1998) 241, hep-th/9803054.

[13] S. Gukov and A. Kapustin, Nucl. Phys. B545 (1999) 283, hep-th/9808175.

[14] A. Gorsky, S. Gukov, and A. Minorov, Nucl. Phys. B517 (1998) 409, hep-th/9707120;
Nucl. Phys. B518 (1998) 689, hep-th/9710239.

[15] S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, hep-th/9505105;

S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, Nucl. Phys. B459 (1996) 537, hep-th/9508155;

M. Bershadsky, V. Sadov, and C. Vafa, Nucl. Phys. B463 (1996) 420, hep-th/9511222;

A. Klemm and P. Mayr, Nucl. Phys. B469 (1996) 37, hep-th/9601014;

S. Katz, D. Morrison, and M. Plesser, Nucl. Phys. B477 (1996) 105, hep-th/9601108;

A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. Warner, Nucl. Phys. B477 (1996) 746, hep-th/9604034;

P. Berglund, S. Katz, A. Klemm, and P. Mayr, Nucl. Phys. B483 (1997) 209, hep-th/9605154;

S. Katz, A. Klemm, and C. Vafa, Nucl. Phys. B497 (1997) 173, hep-th/9609239;

S. Katz, P. Mayr, and C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53, hep-th/9706110;

M. Aganagic and M. Gremm, Nucl. Phys. B524 (1998) 207, hep-th/9712011.

For reviews see: W. Lerche, Nucl. Phys. Proc. Suppl. 55B (1997) 83, Fortsch. Phys. 45 (1997) 293, hep-th/9611190;

A. Klemm, hep-th/9705131;

S. Katz, P. Mayr, and C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53, hep-th/9706110.

[16] T. Yokono, Nucl. Phys. B532 (1998) 210, hep-th/9803123.

[17] K. Landsteiner, J. Pierre, and S. Giddings, Phys. Rev. D55 (1997) 2367, hep-th/9609059;

K. Ito, Phys. Lett. B406 (1997) 54, hep-th/9703180.

[18] E. Martinec and N. Warner, Nucl. Phys. B459 (1996) 97, hep-th/9509161;

E. Martinec, Phys. Lett. B367 (1996) 91, hep-th/9510204.

[19] A. Klemm, W. Lerche, and S. Theisen, Int. J. Mod. Phys. A11 (1996) 1929, hep-th/9505150;

K. Ito and S.-K. Yang, hep-th/9603073;

K. Ito and N. Sasakura, Nucl. Phys. B484 (1997) 141, hep-th/9608054;

J.M. Isidro, A. Mukherjee, J.P. Nunes, and H.J. Schnitzer, Nucl. Phys. B492 (1997) 647, hep-th/9609116; Int. J. Mod. Phys. A13 (1998) 233, hep-th/9703176; Nucl. Phys. B502 (1997) 363, hep-th/9704174;

M. Alishahiha, Phys. Lett. B398 (1997) 100, hep-th/9609157; Phys. Lett. B418 (1998) 317, hep-th/9703186;

Y. Ohta, J. Math. Phys. 40 (1999) 3211, hep-th/9906207;

U. Feichtinger, hep-th/9908143;

J.M. Isidro, J. Geom. Phys. 29 (1999) 334.

[20] E. D'Hoker, I.M. Krichever, and D.H. Phong, Nucl. Phys. B489 (1997) 179, hep-th/9609041.

[21] E. D'Hoker, I.M. Krichever, and D.H. Phong, Nucl. Phys. B489 (1997) 211, hep-th/9609145, Nucl. Phys. B494 (1997) 89, hep-th/9610156;

E. D'Hoker and D.H. Phong, Phys. Lett. B397 (1997) 94, hep-th/9701055; Nucl. Phys. B530 (1998) 537, hep-th/9804124; Nucl. Phys. B530 (1998) 611, hep-th/9804125;

Nucl. Phys. B534 (1998) 697, hep-th/9804126; hep-th/9808156; hep-th/9903002; hep-th/9903068; hep-th/9906027.

[22] E. D'Hoker and D.H. Phong, Nucl. Phys. B513 (1998) 405, hep-th/9709053.

[23] G. Chan and E. D'Hoker, hep-th/9906193.

[24] E. Witten, Surv. Diff. Geom. 1 (1991) 243;

R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59;

B. Dubrovin, hep-th/9407018;

A. Marshakov, A. Minorov, and A. Morozov, Phys. Lett. B389 (1996) 43, hep-th/9607109; Mod. Phys. Lett. A12 (1997) 773, hep-th/9701014; hep-th/9701123;

G. Bonelli and M. Matone, Phys. Rev. D58 (1998) 045006, hep-th/9712025;

G. Bertoldi and M. Matone, Phys. Lett. B425 (1998) 104, hep-th/9712039; Phys. Rev. D57 (1998) 6483, hep-th/9712109;

K. Ito and S.K. Yang, Phys. Lett. B433 (1998) 56, hep-th/9803126;

J.M. Isidro, Nucl. Phys. B539 (1999) 379, hep-th/9805051;

A. Minorov, hep-th/9903088.

[25] M. Matone, Phys. Lett. B357 (1995) 342, hep-th/9506102;

J.D. Edelstein, M. Mariño, and J. Mas, Nucl. Phys. B541 (1999) 671, hep-th/9805172;

J.D. Edelstein and J. Mas, Phys. Lett. B452 (1999) 69, hep-th/9901006; hep-th/9902161;

J.D. Edelstein, M. Gómez-Reino, and J. Mas, hep-th/9904087;

Y. Ohta, J. Math. Phys. 40 (1999) 4089, hep-th/9904121; hep-th/9905126;

M. Mariño, hep-th/9906207 for a review, and additional references.

[26] N. Dorey, V. Khoze, and M. Mattis, Phys. Rev. D54 (1996) 2921, hep-th/9603136; Phys. Lett. B390 (1997) 205, hep-th/9606199; Phys. Lett. B388 (1996) 324, hep-th/9607066; Phys. Rev. D54 (1996) 7832, hep-th/9607202;

F. Fucito and G. Travaglini, Phys. Rev. D55 (1997) 1099, hep-th/9605215;

K. Ito and N. Sasakura, Phys. Lett. B382 (1996) 95, hep-th/9602073; Mod. Phys. Lett. A12 (1997) 205, hep-th/9609104;

H. Aoyama, T. Harano, M. Sato, and S. Wada, Phys. Lett. B388 (1996) 331, hep-th/9607076;

T. Harano and M. Sato, Nucl. Phys. B484 (1997) 167, hep-th/9608060;

Y. Yoshida, hep-th/9610211;

M. Slater, Phys. Lett. B403 (1997) 57, hep-th/9701170;

V. Khoze, M. Mattis, and M. Slater, Nucl. Phys. B536 (1998) 69, hep-th/9804009;

I. Sacho and W. Weir, hep-th/9910015.

[27] See, for example, J. Polchinski, *String Theory, vol. I* (Cambridge University Press, 1998, Cambridge), p. 215.

[28] J. A. Minahan, D. Nemeschansky, and N. P. Warner, Nucl. Phys. B528 (1998) 109, hep-th/9710146.

[29] I. Ennes, C. Lozano, S. Naculich, and H. Schnitzer, in preparation.