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# Microscopic- versus Effective Coupling in N=2 Yang-Mills With Four Flavours

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**Abstract**

We determine the instanton corrections to the effective coupling in $SU(2)$, $N=2$ Yang-Mills theory with four flavours to all orders. Our analysis uses the $S(2,Z)$-invariant curve and the two instanton contribution obtained earlier to fix the higher order contributions uniquely.

# 1 Introduction

Seiberg and Witten [1, 2] proposed exact results for $SU(2)$, $N=2$ supersymmetric Yang-Mills theory with and without matter multiplets. These include, in particular an exact expression for the mass spectrum of BPS-states for these theories. Their solutions also provide a mechanism, based on monopole condensation, for chiral symmetry breaking and confinement in $N=2$ Yang-Mills with and without coupling to fundamental hypermutiplets respectively. The results of Seiberg and Witten have been generalised to a variety of gauge groups [3] describing a number of interesting new phenomena [4].

On another front the low energy effective theories arising from non-Abelian YM-theory have been identified with those describing the low energy dynamics of certain intersecting brane configurations in string theory [5]. The latter approach provides an elegant geometrical representation of the low energy dynamics of strongly coupled supersymmetric gauge theory.

For gauge groups $SU(2)$ and $N_F \leq 3$ massless flavours it has since been shown that the solution [1, 2] are indeed the only ones compatible with supersymmetry and asymptotic freedom in these theories [6]. However, a number of issues have still resisted an exact treatment so far. In particular the precise relation between the low energy effective coupling $\tau_{eff}$ and the microscopic coupling $\tau$ in the scale invariant $N_F=4$ theory has not been understood so far. Indeed, while the two couplings were first assumed to be identical in [2] it was later found by explicit computation that there are, in fact, perturbative and instanton corrections [7]. On the other hand, explicit instanton calculus is so far limited to topological charge $k \leq 2$. A related observation has been made in the D-brane approach to scale invariant theories. The details of the conclusions reached there are, however, somewhat different [8].

The purpose of the present paper is to fill this gap. Our analysis uses a combination of analytic results from the theory of conformal mappings combined with the known results form instanton calculus. More precisely we consider the sequence $\tau \mapsto z \in \mathbf{C} \mapsto \tau_{eff}$. We will then argue that given the Seiberg-Witten curve together with some suitable assumptions on the singular behaviour of the instanton contributions there is a one-parameter family of admissible maps $\tau_{eff} \mapsto \tau$. The remaining free parameter is in turn determined by the two-instanton contribution to the asymptotic expansion at weak coupling. This coefficient has been computed explicitly in [7]. This then determines the map completely. Although we are not able to give a closed form of the map $\tau \mapsto \tau_{eff}$ globally the higher order instanton coefficients can be determined iteratively. We further discuss some global properties of the map qualitatively. In particular we will see that it is not single valued meaning that the instanton corrections lead to a cut in the strong coupling regime. In this note we restrict ourselvs to gauge group $SU(2)$ leaving the extension to higher groups [9] for future work.

# 2 Review of N=2 Yang-Mills with 4 Flavours

To prepare the ground let us first review some of the relevant features [2] of the theory of interest, that is $N=2$ YM-theory with 4 hypermultiplets $Q^r$ and $\tilde{Q}_r$, $r=1,\cdots,4$, in the fundamental representation. In $N=1$ language the hypermultiplets are described by two chiral multiplets containing the left handed quarks and antiquarks respectively. These are in isomorphic representations of the gauge group $SU(2)$. The global symmetry group therefore contains a $SO(8)$ or, more precisely, a $O(8)$ due to invariance under the $\mathbf{Z}_2$ "parity" which exchanges a left handed quark with its antiparticle, $Q_1 \leftrightarrow \tilde{Q}_1$, with all other fields invariant. At the quantum level this $\mathbf{Z}_2$ is anomalous due to the contributions from odd instantons.

We consider the Coulomb branch with constant scalar $\varphi = a \neq 0$ in the $N=2$ vector multiplet. This breaks the gauge group $SU(2) \rightarrow U(1)$. The charged hypermultiplets then have mass $M = \sqrt{2}|a|$ and transform as a vector under $SO(8)$, rather than $O(8)$ due to the $\mathbf{Z}_2$-anomaly. In addition, there are magnetic monopoles solutions leading to 8 fermionic zero modes from the 4 hypermultiplets. These turn the monopoles into spinors of $SO(8)$. The symmetry group is therefore the universal cover of $SO(8)$ or Spin(8) with centre $\mathbf{Z}_2 \times \mathbf{Z}_2$. Following [2] we label the representations $\mathbf{Z}_2 \times \mathbf{Z}_2$ by according to the Spin(8) representations, that is the trivial representation $o$, the vector representation $v$ and the two spinor representations $s$ and $c$. To decide in which spinor representation the monopoles and dyons transform one considers the action of an electric charge rotation $e^{\pi i Q}$ on these states. Here the electric charge $Q$ is normalised such that the massive gauge bosons have charge $\pm 2$. This action is conveniently described by

$$e^{\pi i Q} = e^{i n_m \theta}(-1)^H, \tag{1}$$

where states with even, odd $n_e$ are $(-1)^H$ even, odd respectively. On the other hand, for consistency, the monopole anti-monopole annihilation process requires a correlation between chirality in Spin(8) and electric charge [2]. We therefore identify $(-1)^H$ with the chirality operator in the spinor representations of Spin(8). Hence dyons with even and odd electric charge transform in one or the other spinor representation of Spin(8) respectively.

The outer automorphism $S_3$ of Spin(8) that permutes the three non-trivial representations $v$, $s$ and $c$ is closely connected to the proposed duality group $SL(2,\mathbf{Z})$ of the quantum theory. Indeed there is a homomorphism $SL(2,\mathbf{Z}) \rightarrow S_3$, so that the invariance group of the spectrum is given by the semi direct product Spin(8) and $SL(2,\mathbf{Z})$. The kernel of this homomorphism plays an important part in our analysis below. It consists of the elements in $SL(2,\mathbf{Z})$ that commute with the global symmetry group $SL(2,\mathbf{Z})$. These are the matrices congruent to 1 (mod(2)). They are conjugate to the subgroup $\Gamma_0(2)$. The fundamental domain of this subgroup is the space of inequivalent coupling in the analysis presented below.

One can further formalise this structure in terms of the hyperelliptic curve that controls the low energy behaviour of the model [2]. For this one seeks a curve $y^2 = F(x, u, \tau)$ such that the differential form

$$\omega = \frac{\sqrt{2}}{8\pi}\frac{\mathrm{d}x}{y} \tag{2}$$

has the periods $(\frac{\partial a_D}{\partial u}, \frac{\partial a}{\partial u})$ with $(a_D, a)$ given by [2]

$$a = \sqrt{\frac{u}{2}} \quad \text{and} \quad a_D = \tau_{eff}\, a \tag{3}$$

where $\tau_{eff}$ [7] is the low energy effective coupling whose dependence on the microscopic coupling $\tau$ will be determined below. The correct curve consistent with $SL(2,\mathbf{Z})$ duality is given by [2]

$$y^2 = (x - ue_1(\tau))(x - ue_2(\tau))(x - ue_2(\tau)), \tag{4}$$

where $e_i(\tau)$ are the modular forms corresponding to the three subgroups of $SL(2,\mathbf{Z})$, conjugate to the index 3 subgroup $\Gamma_0(2)$. An equivalent form of the curve is obtained by rescaling $x = x_0 u$, $y = \frac{1}{2} y_0 u^{3/2}$ i.e.

$$y_0^2 = (x_0 - e_1(\tau))(x_0 - e_2(\tau))(x_0 - e_2(\tau)). \tag{5}$$

# 3 Map: $\tau \mapsto \tau_{eff}$

We now have the necessary ingredients to determine the precise relation between $\tau_{eff}$ and $\tau$. We begin with the observation that, according to the structure of the effective theory presented above, the fundamental domain $D_\Gamma$ of any of the three subgroups conjugate to $\Gamma_0(2)$ can be used as the space of inequivalent effective couplings. The three choices are then related by the Spin(8) "triality" relating the 3 different non-trivial representations $v$, $s$ and $c$. Each fundamental domain is described by a triangle in the upper half plane ($\mathrm{Im}(\tau) \geq 0$), bounded by circular arcs [10]. The 3 singularities are conjugate to the points $(i\infty, -1, 1)$ corresponding to the weak coupling regime, massless monopoles and massless dyons with charge $(n_m, n_e) = (1,1)$ mod(2) respectively.

In the absence of perturbative- and instanton corrections the effective coupling is identified with the microscopic coupling $\tau$. This applies to $N = 4$ theories. In the scale invariant $N = 2$ theory considered here the situation is different. As shown in [7], the effective coupling is finitely renormalised at the one loop level and furthermore receives instanton corrections. Some information about the fundamental domain $\bar{D}$ of microscopic couplings $\tau$ is obtained from the following observations:
*a)* As the microscopic coupling does not enter in the mass formula, its fundamental domain is not constrained to be that of a subgroup of $SL(2,\mathbf{Z})$ [6]. Nevertheless we require that the imaginary part of $\tau$ be bounded from below. Correspondingly the different determinations of $\tau$ for a given $\tau_{eff}$ are related by a transformation in $PSL(2,\mathbb{R})$. Hence,

$$\tau \in C[H/G] \quad \text{where} \quad G \subset PSL(2,\mathbb{R}), \tag{6}$$

where $C[\ ]$ denotes a certain covering. This is a domain bounded by circular arcs, conformally equivalent to the punctured 2-sphere, $\mathbf{C} - \{a_1, \cdots, a_n\}$ [10].
*b)* We know of no principle excluding the possibility that the number of vertices of $\bar{D}$ be different from that of $D_\Gamma$. On the other hand such extra singularities have no obvious physical interpretation. We therefore discard this possibility.

Equipped with this information we will now determine the homomorphism that maps $D_\Gamma$ into the fundamental domain of microscopic couplings[1] $\bar{D}$. It follows from general arguments [2, 7] that this map has an expansion of the form

$$\tau_{eff} = \sum_{n=0}^{\infty} c_n q^n \quad \text{where} \quad q = e^{\pi i \tau} \tag{7}$$

[1]As will become clear below the two domains cannot be isomorphic.

The coefficients $c_i$, represent the perturbative one-loop ($i=0$) and instanton ($i>0$) corrections respectively. The contributions from odd instantons to $\tau_{eff}$ vanishes. This is due the fact that the part of the effective action determining the effective coupling is invariant under the $\mathbf{Z}_2$-"parity". The first two non-vanishing coefficients of the expansion (7) are known [7]

$$c_0 = \frac{i}{\pi} 4 \ln 2 \quad \text{and} \quad c_2 = -\frac{i}{\pi} \frac{7}{2^5 \cdot 3^6}. \tag{8}$$

To continue we use some elements of the theory of conformal mappings [10]. That is we consider the maps from the punctured 2-sphere $S = \mathbf{C} - \{a_1, \cdots, a_n\}$ to polygons in the upper half plane, bounded by circular arcs. Concretely we consider the sequence $\tau_{eff} \mapsto z \in S \mapsto \tau$ (see Fig.1). The form of such mappings is generally complicated. However, their Schwarzian derivative takes a remarkably simple form [10]

$$\{\tau, z\} = \sum_i \frac{1}{2} \frac{1-\alpha_i^2}{(z-a_i)^2} + \frac{\beta_i}{z-a_i}. \tag{9}$$

The parameters $\alpha_i$ measure the angles of the polygon in units of $\pi$. The accessory parameters $\beta_i$ do not have a simple geometric interpretation but are determined uniquely up to a $SL(2,\mathbf{C})$ transformation of $S$. Furthermore they satisfy the conditions [10]

$$\sum_{i=1}^{n} \beta_i = 0 \quad , \quad \sum_{i=1}^{n} \left[2 a_i \beta_i + 1 - \alpha_i^2\right] = 0,$$

$$\sum_{i=1}^{n} \left[\beta_i a_i^2 + a_i \left(1 - \alpha_i^2\right)\right] = 0 \tag{10}$$

In the present situation it is convenient to orient the polygons such that they have a vertex at infinity with zero angle (see Fig. 1 ) corresponding to the weak coupling singularity ($\tau = \tau_{eff} = i\infty$). The above conditions then simplify to

$$\sum_{i=1}^{\infty} \beta_i = 0 \tag{11}$$

$$\sum_{i=1}^{n-1} \left(2 a_i \beta_i - \alpha_i^2\right) = (2-n).$$

As explained at the beginning of this section, in the case at hand, the polygon on the $\tau_{eff}$-side corresponds to the fundamental domain of $\Gamma_0(2)$. The corresponding parameters $(a_i, \alpha_i, \beta_i)$ are given by [6]

$$a_1 = a_{-1} = -1, \qquad \beta_1 = \beta_{-1} = -\frac{1}{4}, \qquad \alpha_i = 0. \tag{12}$$

The parameters for the polygon on the $\tau$-side, $(\tilde{a}_i, \tilde{\alpha}_i, \tilde{\beta}_i)$ are to be determined. However, the conditions (11) together with the symmetry $\tau \to -\bar{\tau}$ leaves only one free parameter. Indeed, without restricting the generality we can choose $\tilde{a}_i = a_i$. Furthermore $\tilde{\alpha}_i = -\tilde{\alpha}_i$. Then (11) implies

$$\tilde{\beta}_{-1} = -\tilde{\beta}_1 = \frac{1}{4}\left(1 - 2\tilde{\alpha}_1{}^2\right) \tag{13}$$

leaving only one parameter, $\tilde{\alpha}_1$ say, undetermined. As we shall now see this parameter is in turn determined by the two instanton contribution in (7). For this we make use of the identity

$$\begin{aligned}\{\tau,\tau_{eff}\}\left(\frac{\partial\tau_{eff}}{\partial z}\right)^2 &= \{\tau,z\}-\{\tau_{eff},z\} \qquad (14)\\ &= \sum_{i=1}^{\tilde{n}-1}\frac{1}{2}\frac{1-\tilde{\alpha}_i^2}{(z-\tilde{a}_i)^2}+\frac{\tilde{\beta}_i}{(z-\tilde{a}_i)}-\sum_{i=1}^{n-1}\frac{1}{2}\frac{1-\alpha_i^2}{(z-a_i)^2}+\frac{\beta_i}{(z-a_i)}.\end{aligned}$$

To continue we invert (7) as

$$\tau=\tau_{eff}-c_0-c_2e^{-2\pi ic_0}q_{eff}+\left(2\pi ic_2^2-c_4\right)e^{-4\pi ic_0}q_{eff}^2+O(q_{eff}^4). \qquad (15)$$

Finally we need the form of $\tau_{eff}(z)$, that is the inverse modular function for $\Gamma_0(2)$ [11, 6]

$$\tau_{eff}=i\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{-1+z}{1+z}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2}{1+z}\right)} \qquad (16)$$

This function has the asymptotic expansion for large $z$

$$\tau_{eff}(z)=\frac{i}{\pi}\left[3\ln 2+\ln z-\frac{5}{16}\frac{1}{z^2}\right]+O(z^{-3}). \qquad (17)$$

Substituting (17) into the right hand side of (14) we end up with

$$-\{\tau,\tau_{eff}\}\left(\frac{\partial\tau_{eff}}{\partial z}\right)^2=\frac{7}{2\cdot 3^5}\frac{1}{z^4}+\left(\frac{7\cdot 7213}{2^5\cdot 3^{10}}+2^{10}i\pi c_4\right)\frac{1}{z^6} \qquad (18)$$

Substitution of (18) into (14) then leads to

$$\tilde{\alpha}_1{}^2=\frac{7}{2^2\cdot 3^5} \qquad (19)$$

which then fixes the erstwhile free parameter in $\{\tau,z\}$. This is the result we have been aiming at. Indeed all higher instanton coefficients are now determined implicitly by the equation

$$\{\tau,\tau_{eff}\}=\left(\frac{\partial z}{\partial\tau_{eff}}\right)^2\left[\{\tau,z\}-\{\tau_{eff},z\}\right]. \qquad (20)$$

In order to integrate (14) one notices [11] that any solution of (9) can be written as a quotient

$$\tau(z)=\frac{(u_1(z)+du_2(z))}{(eu_1(z)+fu_2(z))}, \qquad (21)$$

where $u_1,u_2$ are two linearly independent solutions of the hypergeometric differential equation

$$(1+z)(1-z)\frac{d^2}{dz^2}u(z)+((c-2)z+c-2a-2b)\frac{d}{dz}u(z)+\frac{2ab}{1+z}=0, \qquad (22)$$

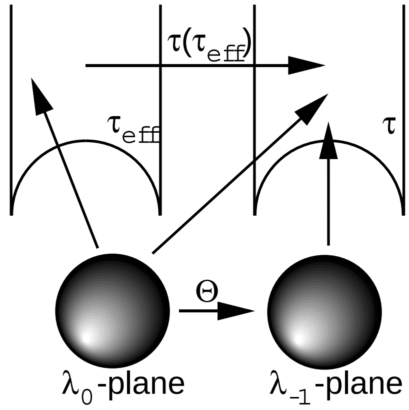


Figure 1:

with

$$c = 1, \quad ; \quad b(c-a) + a(c-b) = \frac{1}{2} \quad \text{and} \quad (a-b)^2 = \tilde{\alpha}_1^2. \tag{23}$$

The coefficients $d, e, f$ are in turn determined by the asymptotic expansion

$$\tau(z) = \frac{i}{\pi} \ln \frac{z}{2} + \frac{i}{8\pi} z^{-2} \left( -\frac{5}{2} + \frac{7}{3^6} \right) + O(z^{-4}). \tag{24}$$

Substitution of $z$ in (21) by

$$z(\tau_{eff}) \equiv \lambda_{-1} = \frac{2}{\lambda_0(\tau_{eff})} - 1, \tag{25}$$

where $\lambda_0$ is the automorphic function

$$\begin{aligned} \tau_{eff}^{-1} \quad &: \quad \tau_{eff} \mapsto \lambda_0 \in \mathbf{C}, \qquad (26) \\ \lambda_0(\tau_{eff}) \quad &= \quad 16 q_e \prod_{n=1}^{\infty} \left( \frac{1+q_e^{2n}}{1+q_e^{2n-1}} \right)^8 \qquad \text{with} \qquad q_e = \exp(i\pi\tau_{eff}), \end{aligned}$$

then integrates (14) (Fig. 1). To extract the instanton coefficients one needs to invert the map $\tau_{eff}(\tau)$. We have done this to $O(q^4)$ allowing us to predict the 4-instanton coefficient

$$c_4 = \frac{i}{\pi} \frac{7 \cdot 17 \cdot 421}{2^6 \cdot 3^{10} \cdot 521}. \tag{27}$$

We close with the observation that globally the inverse function $\tau_{eff}(\tau)$ cannot be single valued. Indeed, existence of a single valued inverse function $z(\tau)$ requires $\tilde{\alpha}_1 = p$ or $\alpha = 1/p$, $p \in \mathbf{Z}$ [11]. The physical interpretation is that the instanton corrected effective coupling $\tau_{eff}(\tau)$ has a cut somewhere in the strong coupling region. It would be interesting to understand this property from the physics side.

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